Laplace transforms

- Once again a practical exposition, not fully mathematically rigorous
- Definition

$F(s) = \int_0^\infty f(t).e^{-st}.dt$ NB lower limit of integral = 0 **unilateral LT**

more rigorously $F(s) = \int_{0+}^\infty f(t).e^{-st}.dt = {}_{\text{limit } h\to 0}\int_{|h|}^\infty f(t).e^{-st}.dt$

[Another variant exists $F(s) = \int_{-\infty}^\infty f(t).e^{-st}.dt$ **bilateral LT]**

Unilateral LT convenient for systems where nothing happens before t=0

the inverse Laplace transform is much more complicated mathematically than the Fourier transform,

$f(t) = (1/2\pi j)\int_{c-j\infty}^{c+j\infty} F(s).e^{st}.ds$ $j = \sqrt{-1}$

Cauchy principal value of integral in complex plane

However, this is not generally required in most practical cases. There are many problems where inverse transforms can be found by inspection.

Conventions

- \- as for Fourier

  f: function to be transformed

  F: Laplace transform of f $F = LT[f]$ and inverse $f = LT^{-1}[F]$

Unless specifically stated all functions f(t) are assumed to take the value

$$f(t) = 0 \quad t < 0$$

not a real constraint for practical problems

Formally, this can always be achieved for any function by multiplying by unit step function u(t)

- Why use the Laplace transform instead of Fourier?

  particularly suited for transient problems

  some functions don't converge

  Fourier response is an integral

  sometimes Laplace vs Fourier is just preference

The meaning of s

- In Fourier transforms the complementary variable usually has a clear physical meaning,

  eg if working in time t <=> ω or f

  diffraction in optics, where FTs are used, has a similar relationship between spatial distributions and spatial freqency

- Although Laplace transforms look very similar (and many results can be easily obtained by following methods for deriving FTs), the complementary variable s does not have the same physical significance.

  It is a mathematical method of solving problems using transforms

- Since we spent a significant time on the FT, I will not spend so much time on the details of deriving LTs

  integrals are usually straightforward

  I will discuss only transforms we will need here

Some theorems (compare to FT)

PROVE THEM!!

- Linearity $LT[a.f(t)+b.g(t)] = a.F(s) + b.G(s)$

- Shifting in time

  $LT[f(t-\Delta t)] = \int_0^\infty f(t-\Delta t).\, e^{-st}.dt = e^{-s\Delta t} F(s)$

- Translation in s

  $LT[f(t)e^{-at}] = \int_0^\infty f(t)\, e^{-at} e^{-st} dt = F(s+a)$

- Convolution

  $LT[x(t)*y(t)] = X(s)Y(s)$

- Differentiation

  $f'(t) = d/dt\{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} F(s).e^{st}.ds\} = \{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} sF(s).e^{st}.ds\}$

  $LT[f'(t)] = sF(s)$

- Integration

  $\int_0^t f(t)dt = \int_0^t \{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} F(s).e^{st}.ds\}dt$

  $= \{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} (1/s)F(s).e^{st}.ds\}$

  $LT[\int_0^t f(t)dt] = F(s)/s$

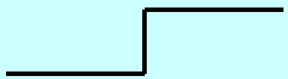
these are results to be remembered (or derived)

Some examples

PROVE THEM!!

(1) $f(t) = e^{-at} \quad t \geq 0$ $\quad F(s) = \int_0^\infty e^{-at}.e^{-st}.dt = \int_0^\infty e^{-(s+a)t}.dt = 1/(s+a)$

(2) $f(t) = u(t) = 1 \quad t \geq 0$ $\quad F(s) = \dfrac{1}{s}$



(3) $f(t) = \delta(t-t_0)$ $\quad F(s) = \int_0^\infty \delta(t-t_0).e^{-st}.dt = e^{-st_0}$ $\quad LT[\delta(t)] = 1$

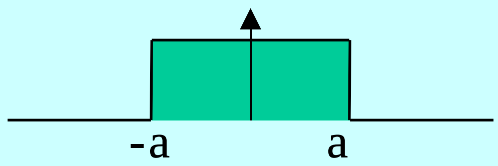
(4) $f(t) = \delta'(t-t_0)$ $\quad F(s) = se^{-st_0}$ $\quad LT[\delta'(t)] = s$

(5) $f(t) = 1-e^{-at}$ $\quad F(s) = \dfrac{a}{s(s+a)}$

(6) $f(t) = ate^{-at}$ $\quad F(s) = \dfrac{a}{(s+a)^2}$

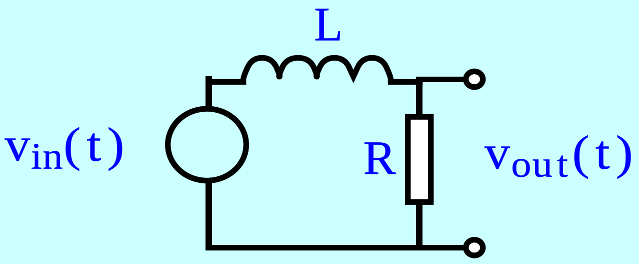
(7) $f(t) = t^n e^{-at}$ $\quad F(s) = \dfrac{n!}{(s+a)^{n+1}}$

(8) $\Pi(t)$ $\quad F(s) = 2\sinh(sa)/s$



Problem solving with LT

- Inductor - resistor circuit



$$v_{out}(t) = i(t)R \qquad L\frac{di}{dt}(t) + Ri(t) = v_{in}(t)$$

$$\frac{L}{R}\frac{dv_{out}}{dt}(t) + v_{out}(t) = v_{in}(t)$$

- Take Laplace transform

$$\frac{L}{R} sV_{out}(s) + V_{out}(s) = V_{in}(s)$$

- solution

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{\frac{sL}{R} + 1} = \frac{a}{s + a} \qquad a = R/L$$

- Example

$v_{in}(t) = u(t) = $ unit step $\qquad V_{in}(s) = \frac{1}{s} \qquad V_{out}(s) = \frac{a}{s(s + a)}$

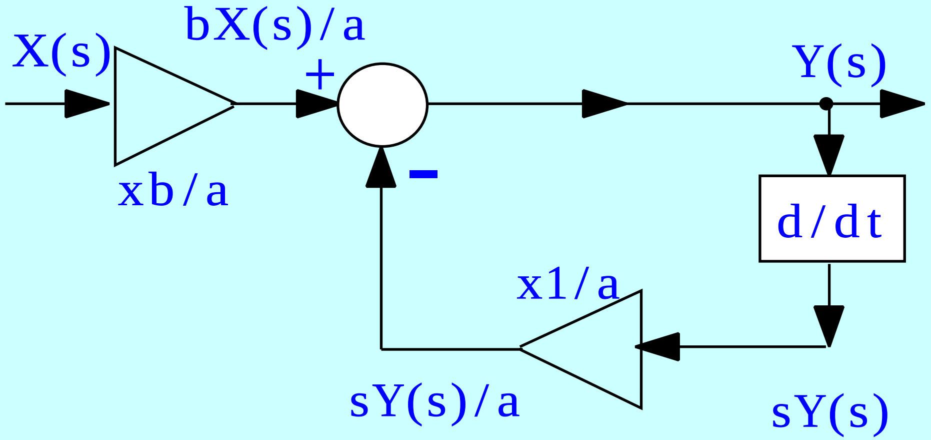
LT of $1 - e^{-at} = v_{out}(t)$

Solution of differential equations

- Solve $\frac{dy(t)}{dt} + ay(t) = bx(t)$ where x(t) = input y(t) = output

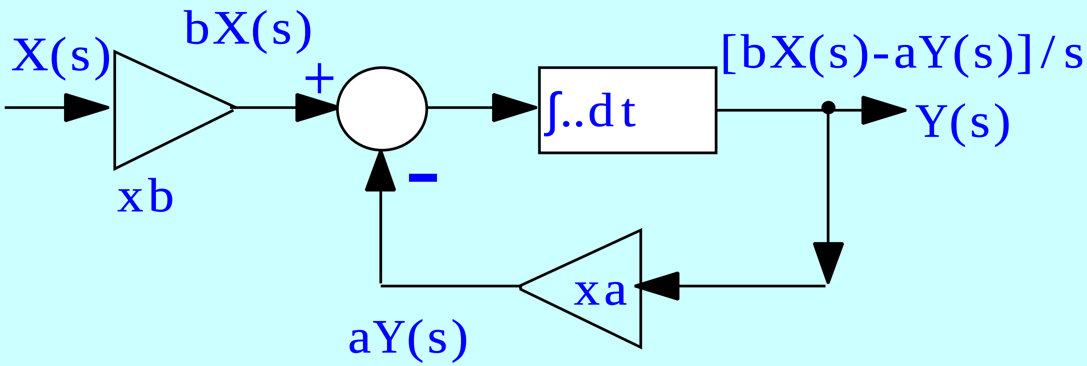
- rewrite as $y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$ and $Y(s) = -\frac{1}{a}sY(s) + \frac{b}{a}X(s)$

- system block diagram



- alternatively

$$y(t) = \int_0^t [bx(u) - ay(u)]du$$



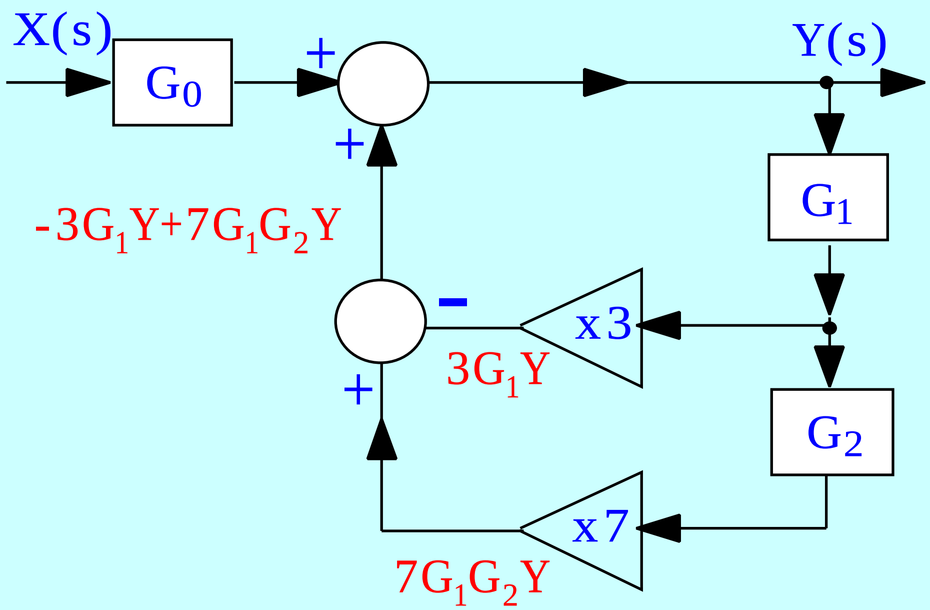
- if x(t) is known, full solution to system response can be found

Example (from 2001 exam)

•(i) derive system transfer function

$$Y = G_0X - 3G_1Y + 7G_1G_2Y$$

$$Y(s) = \frac{G_0X(s)}{1 + 3G_1 - 7G_1G_2}$$



•(ii) G_0 has time domain response $24te^{-2t}$

G_1 is unity gain differentiator

G_2 is unity gain integrator

$$G_0(s) = \frac{24}{(s+2)^2} \qquad G_1(s) = s \qquad G_2(s) = \frac{1}{s}$$

$$Y(s) = \frac{24X(s)}{(s+2)^2(1+3s-7)} = \frac{8X(s)}{(s+2)^2(s-2)}$$

•(iii) Is system stable to small perturbations?

•(iv) Find time domain response to step u(t), for t > 0