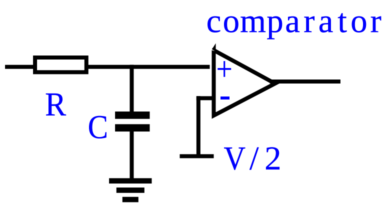
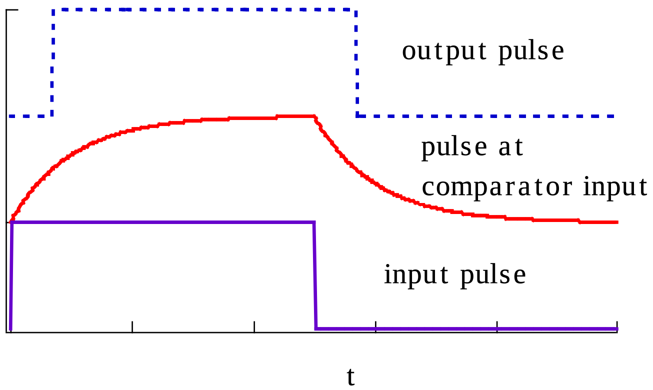
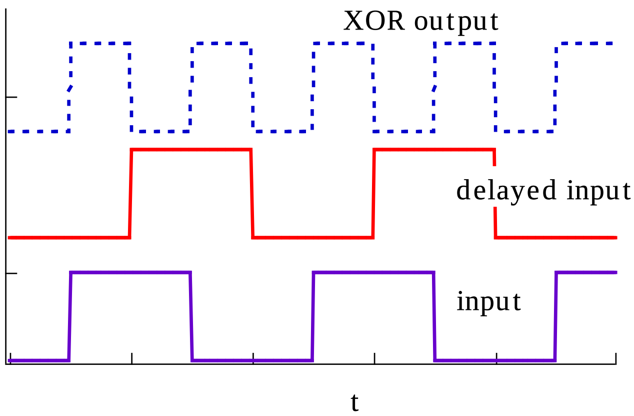
Answers 6

(1) The figure on the right shows a possible circuit. The comparator threshold is set to half the maximum amplitude of the input signal. The signals at different points in the circuit should look as shown in the figure below.





If the clock and the delayed clock are applied to an XOR input the output is as shown. If the delay is T and the original clock has period 4T, the resulting waveform is a clock with period 2T.



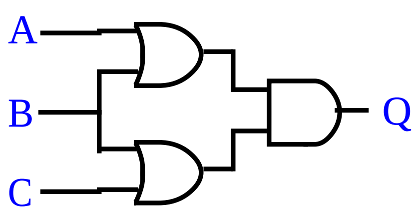
(2) You can make truth tables or rely on algebraic logic, once you have a few results.

A(B+C) = AB + AC	
A + AB = A	AB = A or 0, but A+0 = A = A + A
A + BC = (A+B)(A+C)	use the previous results
AA' = 0	either A or A' must = 0.

and DeMorgan's theorems:

(A+B)' = A'.B' and (AB)' = A' + B'

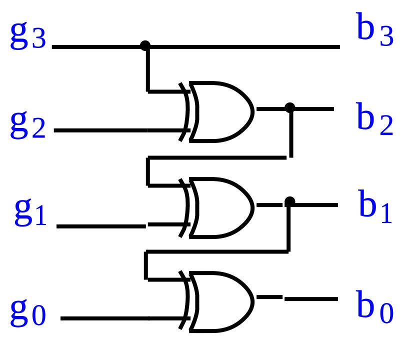
(3) The truth table can be deduced from the logic identities, or from the diagram.
The intermediate results are A+B and B+C so
Q = (A+B)(B+C) = B + AC



The table below shows the result from both methods.

A	B	C	A+B	B+C	Q	AC	B+AC
0	0	0	0	0	**0**	0	**0**
0	0	1	0	1	**0**	0	**0**
0	1	0	1	1	**1**	0	**1**
0	1	1	1	1	**1**	0	**1**
1	0	0	1	0	**0**	0	**0**
1	0	1	1	1	**1**	1	**1**
1	1	0	1	1	**1**	0	**1**
1	1	1	1	1	**1**	1	**1**

(4) Start with the table of codes. The full 4-bit binary to Gray code table is given below. The 3-bit table is just the first 8 entries. The two upper bits can easily be recognised as $b_3 = g_3$ and $b_2 = \mathrm{XOR}(g_3,g_2)$. I got the lower two by inspection - and patience. There's probably a simpler way of finding the lower bits but I haven't spotted it. The figure shows the 4-bit logic. Just drop the lowest gate for the 3-bit case.



decimal	binary	Gray
0	0000	0000
1	0001	0001
2	0010	0011
3	0011	0010
4	0100	0110
5	0101	0111
6	0110	0101
7	0111	0100
8	1000	1100
9	1001	1101
10	1010	1111
11	1011	1110
12	1100	1010
13	1101	1011
14	1110	1001
15	1111	1000

(5) The equation in the s-domain is

$$s^2X(s) - a^2X(s) = F(s)$$

which can be written

$$X(s) = \frac{F(s)}{(s^2 - a^2)} = \frac{F(s)}{2a}\left[\frac{1}{(s-a)} - \frac{1}{(s+a)}\right]$$

so the solution is

$$x(t) = \frac{f(t)}{2a} \quad (e^{at} - e^{-at})$$

where the symbol denotes convolution.

(6) The amplifier impulse response is $f(t) = te^{-t}$

a)The transfer function of a single amplifier is then $F(s) = \frac{1}{(s+1)^2}$

b) The transfer function of two amplifiers is $F_{total}(s) = F_1(s)F_2(s) = F(s)^2 = \frac{1}{(s+1)^4}$

c) The transform of a step u(t) = 1/s, so the transform of the output g(t) is the product of the input function and the overall transfer function, so

$$G(s) = \frac{1}{s}F_{total}(s) = \frac{1}{s(s+1)^4} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3} + \frac{E}{(s+1)^4}$$

To derive the constants, proceed as shown in the lecture to find

A = 1, E = D = C = B = -1

(When differentiating for terms like B and C remember to include the factor which comes from multiple differentiations, eg $B = \frac{1}{3!}\frac{d^3}{ds^3}(\frac{1}{s})$)

then

$$G(s) = \frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3} - \frac{1}{(s+1)^4}$$

Using a result from the lectures which you can easily prove $LT[t^n e^{-t}] = \frac{n!}{(s+1)^{n+1}}$

we find the system response to be $g(t) = u(t) - e^{-t} - te^{-t} - \frac{1}{2}t^2e^{-t} - \frac{1}{6}t^3e^{-t}$

(7) a) For times $t \le 0$ $i(t) = \frac{V}{R}$

b) Eventually, after the opening of the switch, the same current flows as in (a).

c)

$$V - (L_1 + L_2)\frac{di}{dt}(t) = i(t)R$$

$$V - (L_1 + L_2)sI(s) = I(s)R$$

where $L = L_1 + L_2$

$$I(s) = \frac{V}{R + s(L_1 + L_2)} = \frac{V}{L(\frac{R}{L} + s)}$$

So

$$i(t) = \frac{V}{(L_1 + L_2)}e^{-Rt/L} + i(0) = \frac{V}{(L_1 + L_2)}e^{-Rt/L} + \frac{V}{R}$$ for $t > 0$

(8) Before the switch is opened, as before for $t \le 0$, $i(t) = \frac{V}{R}$

The equations after the switch is opened are

$$V - L\frac{di}{dt}(t) = i(t)R + \frac{Q(t)}{C} = i(t)R + \frac{\int_0^t i(u)du}{C}$$

$$V - LsI(s) = I(s)R + \frac{I(s)}{sC}$$

or

$$s^2 LCI(s) + sI(s)RC + I(s) = sCV$$

$$I(s) = \frac{sCV}{s^2 LC + sRC + 1} = \frac{V}{L}\left[\frac{A}{(s-a)} + \frac{B}{(s-b)}\right]$$

the values of a and b are

$$a,b = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The current in the system can be written as

$$i(t) = (V/L)[Ae^{at} + Be^{bt}] + C$$

with A = a/(a-b) B = -b/(a-b) and C = i(0) = V/R

The system is stable if both poles are in the left half plane. This requires that the square root term in a or b should be smaller in magnitude than the first term. This will always be the case, but complex values of a or b mean there is a damped oscillatory solution, as you would expect.

(9) In the s-domain, the system response is

$$Y(s) = G_0(s)G_1(s)G_2(s)X(s)$$

Since x(t) is a δ impulse, X(s) = 1. The other transfer functions are

$$G_0(s) = \frac{1}{s} \qquad G_1(s) = \frac{s\tau_1}{(1 + s\tau_1)} = \frac{s}{(s+2)} \qquad G_2(s) = \frac{1}{(1 + s\tau_2)} = \frac{3}{(s+3)}$$

$$Y(s) = \frac{3s}{s(s+2)(s+3)} = \frac{3}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$$

$$Y(s) = \frac{3}{(s+2)} - \frac{3}{(s+3)}$$

so $\quad y(t) = 3(e^{-2t} - e^{-3t})$

(10)The system response is given by the equation

$$A[x(t) + By(t-T)] = y(t)$$

Laplace transforming, we find

$$A[X(s) + Be^{-sT}Y(s)] = Y(s)$$

so

$$Y(s) = \frac{AX(s)}{(1 - ABe^{-sT})}$$

We don't know what value of T will apply (imagine a large auditorium) and we would like the system to be stable for all T values. This is somewhat different from the case shown in the lectures, since there is no simple pole. However, clearly Y(s) should always remain finite for stability. Since $e^{-sT} < 1$, we can see that, provided $AB < 1$, this condition can be achieved. This should be possible provided there are no nasty phase shifts associated with reflections or other such phenomena.

(11) $$f(t) = \frac{t}{\tau} e^{-t/\tau} = \frac{n\Delta t}{\tau} e^{-n\Delta t/\tau} = nae^{-na}$$

with $a = \Delta t/\tau$. Ignoring the factor scaling factor a in front, $f_n = ne^{-na}$ and

$$F(z) = e^{-a}z^{-1} + 2e^{-2a}z^{-2} + 3e^{-3a}z^{-3} + 4e^{-4a}z^{-4\dots} \dots + ne^{-na}z^{-n} + \dots$$

define $x = e^{-a}z^{-1}$

$$F(z) = x + 2x^2 + 3x^3 + 4x^4 + \dots nx^n + \dots$$

then, rearranging the sum into separate geometrical series which can be summed independently:

$$\begin{aligned} F(z) = x + x^2 + x^3 + x^4 + \dots x^n + \dots \quad &= \mathbf{x/(1-x)} \\ + x^2 + x^3 + x^4 + \dots x^n + \dots \quad &= \mathbf{x^2/(1-x)} \\ + x^3 + x^4 + \dots x^n + \dots \quad &= \mathbf{x^3/(1-x)} \\ + x^4 + \dots x^n + \dots \quad &= \mathbf{x^4/(1-x)} \end{aligned}$$

so

$$F(z) = (x + x^2 + x^3 + x^4 + \dots x^n + \dots)/(1-x) \quad = \mathbf{x/(1-x)^2}$$

ie

$$F(z) = \frac{e^{-a}z^{-1}}{(1-e^{-a}z^{-1})^2}$$

so

$$F^{-1}(z) = \frac{(1-e^{-a}z^{-1})^2}{e^{-a}z^{-1}} = \frac{(1-2e^{-a}z^{-1} + e^{-2a}z^{-2})}{e^{-a}z^{-1}} = e^a z - 2 + e^{-a}z^{-1}$$

$$g_n = e^a f_{n+1} - 2f_n + e^{-a} f_{n-1}$$

with the three weights being the coefficients of the terms. The constant a ignored is only a scaling factor which is not important, unless we are interested in the amplitude after the deconvolution operation.