Boris Push with Spatial Stepping

G. Penn
UC Berkeley / LBNL CBP

July, 2002
NuFact 02
Imperial College, London

Outline

- Boris particle push
- Spatial Boris Push Equations
- advantages of Boris push
- Stability and Accuracy
- Comparison with Runge-Kutta

Boris Push

The Boris push is commonly used in plasma physics simulations because of its speed and stability.

It uses a leapfrog scheme, where the particle is moved, then half of the energy changed is applied.

After this, the momentum is rotated by the B field, and the rest of the energy change applied.

The rotation is automatically energy conserving, and the algorithm is symmetric to time reversal, which improves the performance.

Second order accurate, but requires only one field evaluation per time step.

The Boris push is rarely used for accelerator simulations, because it is more convenient to propagate particles in **z**.

Spatial Boris Push

In collab. with P. Stoltz, J. Cary, at Tech-X (& U. Colorado, Boulder)

The spatial Boris scheme exchanges U for p_z and t for z

First, we replace the equation for pz with the equation for U:

$$\frac{dp_x}{dt} = q(E_x + v_y B_z - v_z B_y)$$
$$\frac{dp_y}{dt} = q(E_y + v_z B_x - v_x B_z)$$
$$\frac{d(U/c)}{dt} = q(E_x v_x + E_y v_y + E_z v_z)$$

Replacing t with z, the governing equations of the spatial Boris scheme are:

$$\frac{dp_x}{dz} = \frac{1}{v_z}\frac{dp_x}{dt} = q\left(\frac{E_x}{v_z} + \frac{v_y B_z}{v_z} - B_y\right)$$
$$\frac{dp_y}{dz} = \frac{1}{v_z}\frac{dp_y}{dt} = q\left(\frac{E_y}{v_z} - \frac{v_x B_z}{v_z} + B_y\right)$$
$$\frac{dU/c}{dz} = \frac{1}{v_z}\frac{dU/c}{dt} = q\left(\frac{v_x E_x}{v_z c} + \frac{v_y E_y}{v_z c} + \frac{E_z}{c}\right)$$

Decomposition (1)

Muon Collaboration

For spatial Boris push, the equations separate into terms that directly change p_z and terms that don't

– In the temporal Boris scheme, the separation is into one piece that changes U (E-fields) and one that doesn't (B-fields)

The terms that directly change p_z are E_z, B_x, and B_y

$$\frac{d}{dz}\begin{pmatrix} p_x \\ p_y \\ U/c \end{pmatrix} = \frac{q}{p_z}\begin{pmatrix} 0 & B_z & E_x/c \\ -B_z & 0 & E_y/c \\ E_x/c & E_y/c & 0 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ U/c \end{pmatrix} + q\begin{pmatrix} -B_y \\ B_x \\ E_z/c \end{pmatrix}$$

For simplicity, rewrite:

$$\frac{dw}{dz} = Mw + b$$

Decomposition (2)

The Boris scheme integrates vector and matrix terms separately

The Boris scheme says first push the vector term one-half step:

$$w^- = w^n + \frac{\Delta z}{2} b$$

This step is implicit and requires some more massaging

Then push the matrix term a full step:

$$w^+ - w^- = M\left(\frac{w^+ + w^-}{2}\right)\Delta z$$

Finally, push the vector term the final half step:

$$w^{n+1} = w^+ + \frac{\Delta z}{2} b$$

Explicit Expression for Boris Push

Muon Collaboration

Step-centered push of matrix term is 2nd-order accurate

Because M is constant, a step-centered scheme will be 2nd-order accurate

$$w^+ - w^- = M\left(\frac{w^+ + w^-}{2}\right)\Delta z$$

Solving for w^+ gives:

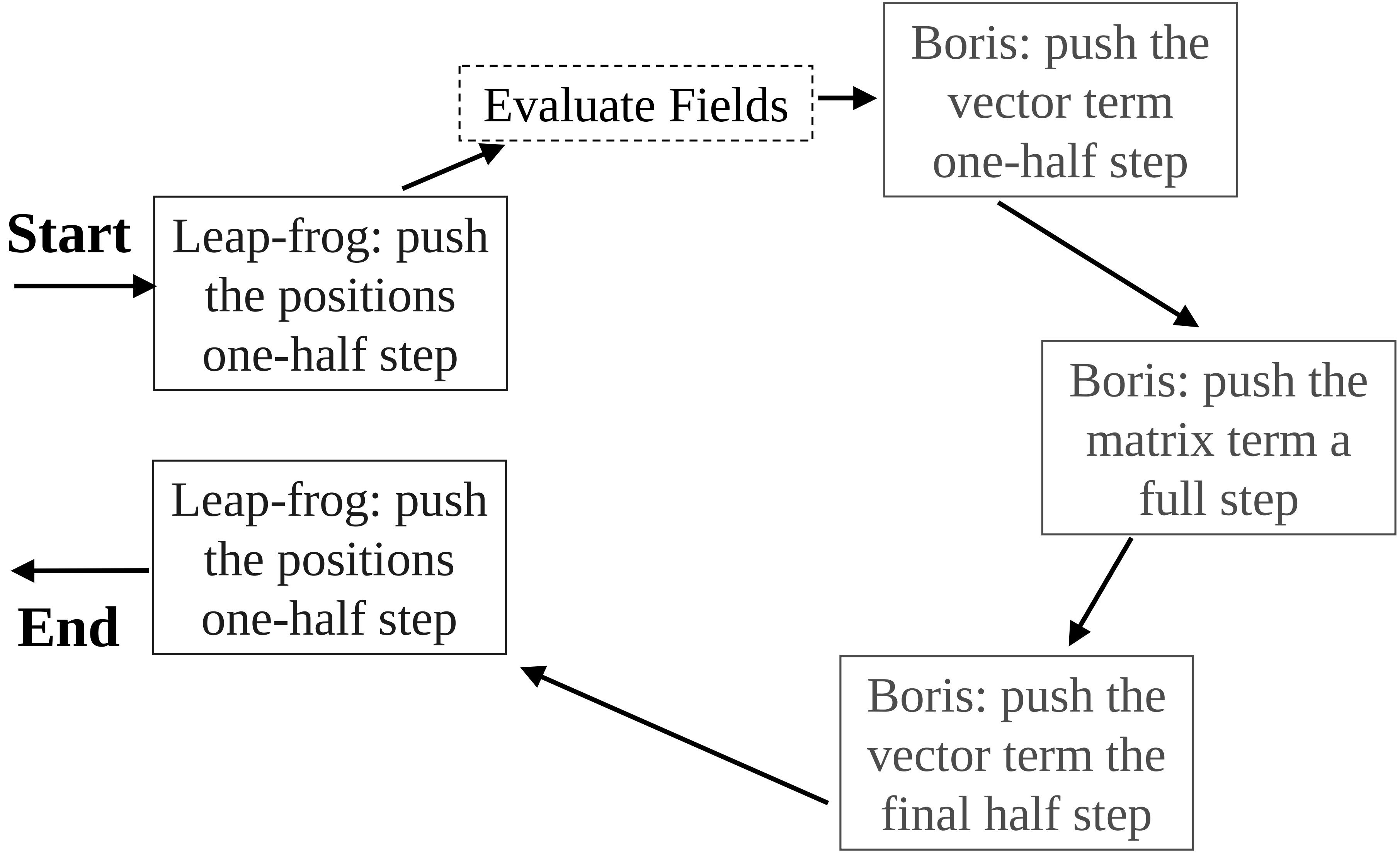
$$w^+ = \left(I - M\frac{\Delta z}{2}\right)^{-1}\left(I + M\frac{\Delta z}{2}\right)w^-$$

$$= (I+R)\, w^-,$$

$$R = \frac{q\Delta z/p_z}{1 + \frac{q^2\Delta z^2}{4p_z^2}\left(B_z^2 - \frac{E_x^2+E_y^2}{c^2}\right)}\begin{pmatrix} \frac{q\Delta z}{2p_z}\left(\frac{E_x^2}{c^2} - B_z^2\right) & B_z + \frac{q\Delta z}{2p_z}\frac{E_x E_y}{c^2} & \frac{E_x}{c} + \frac{q\Delta z}{2p_z}\frac{B_z E_y}{c} \\ -B_z + \frac{q\Delta z}{2p_z}\frac{E_x E_y}{c^2} & \frac{q\Delta z}{2p_z}\left(\frac{E_y^2}{c^2} - B_z^2\right) & \frac{E_y}{c} - \frac{q\Delta z}{2p_z}\frac{B_z E_x}{c} \\ \frac{E_x}{c} + \frac{q\Delta z}{2p_z}\frac{B_z E_y}{c} & \frac{E_y}{c} + \frac{q\Delta z}{2p_z}\frac{B_z E_x}{c} & \frac{q\Delta z}{2p_z}\left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2}\right) \end{pmatrix}$$

Integration Cycle

Muon Collaboration



Boris Push speeds up particle tracking

Muon Collaboration

Spatial motion is calculated a 1/2 step off from momentum/energy evolution:

- typically, use leap-frog
- in ICOOL, split into two half-steps,
  before and after evolution of *w*.

Fields are only evaluated once, where momentum kick is applied

- compared to 4 field evaluations for RK
- local effect on particle, so almost indep. of coordinates
- track as if no field, then replace Δz with separation between planes
  (now a function of transverse co-ords)

As in RK, assumes small energy loss per step.

Boris Push is Space-Symmetric

– except ionization energy loss, scatter, are applied at end of step

Second-order conservation of energy

- also canonical momentum when applicable
- robust for large stepsizes

Errors tend to average out

- in RK scheme, errors will slowly accumulate

Both schemes work well for small phase advances, but Boris push is simpler to calculate

- especially if field calculations are expensive
- less savings for curvilinear (where even field-free is complicated)

Error Scaling

Comparison of Runge-Kutta with Boris, for conserved quantities:

Runge Kutta is approx fifth order accurate (special case, solenoid?)

tends to yield slowly increasing errors.

Boris push only second order, but preserves invariants

maximum error, ϵ, reached in 1 betatron oscillation (or cell period)

Relevant length scale is L, step size is Δ

RK: $\epsilon \sim R_0 \ (\Delta_R / L)^5 \ z / L$ note exponent = 5, not 4

Boris: max $\epsilon \sim B_0 \ (\Delta_B / L)^2$

with 4 x more field calculations per step for RK,

Boris is faster when acceptable to have $\epsilon > B_0^{5/3} \ R_0^{-2/3} \ (L / z)^{2/3}$

Examples

10 T Uniform Solenoid – conserves $P_\perp$

40 cm for π phase advance: after 400 m, crossover point is

$\epsilon \sim 2 \times 10^{-5}$

well below fluctuations for an ensemble of ~ 10^6 particles

FOFO lattice:

1 m half-period with B=2 T, P=200 MeV/c

well above cutoff momentum

after 400 meters, crossover point is roughly the same, $\epsilon \sim 3 \times 10^{-5}$

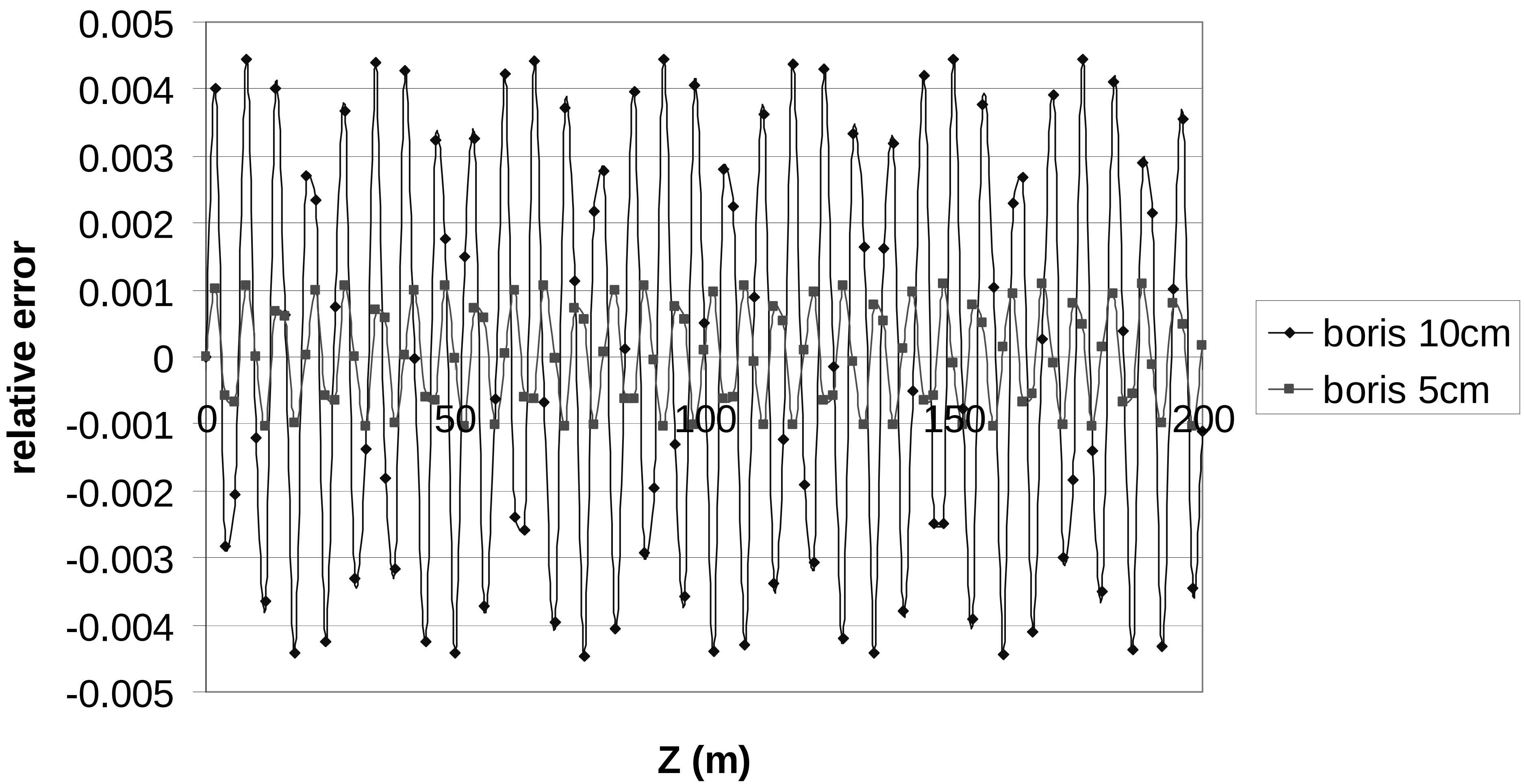
For minimum step sizes fixed by other concerns (e.g., scattering),

Boris step is 4 x faster and may still be more accurate

Boris push errors

Muon Collaboration

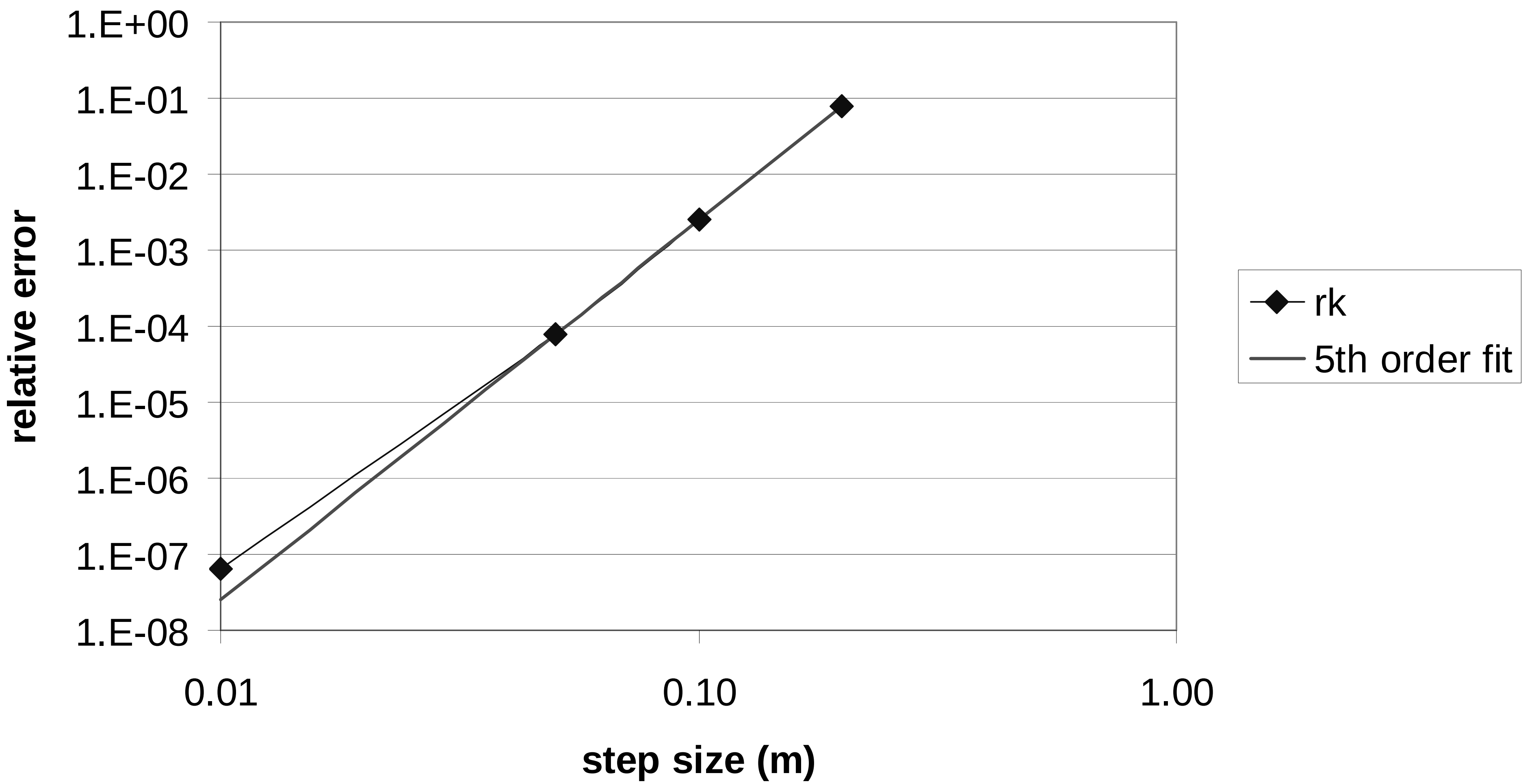
FOFO: relative error in angular momentum



Runge Kutta errors

Muon Collaboration

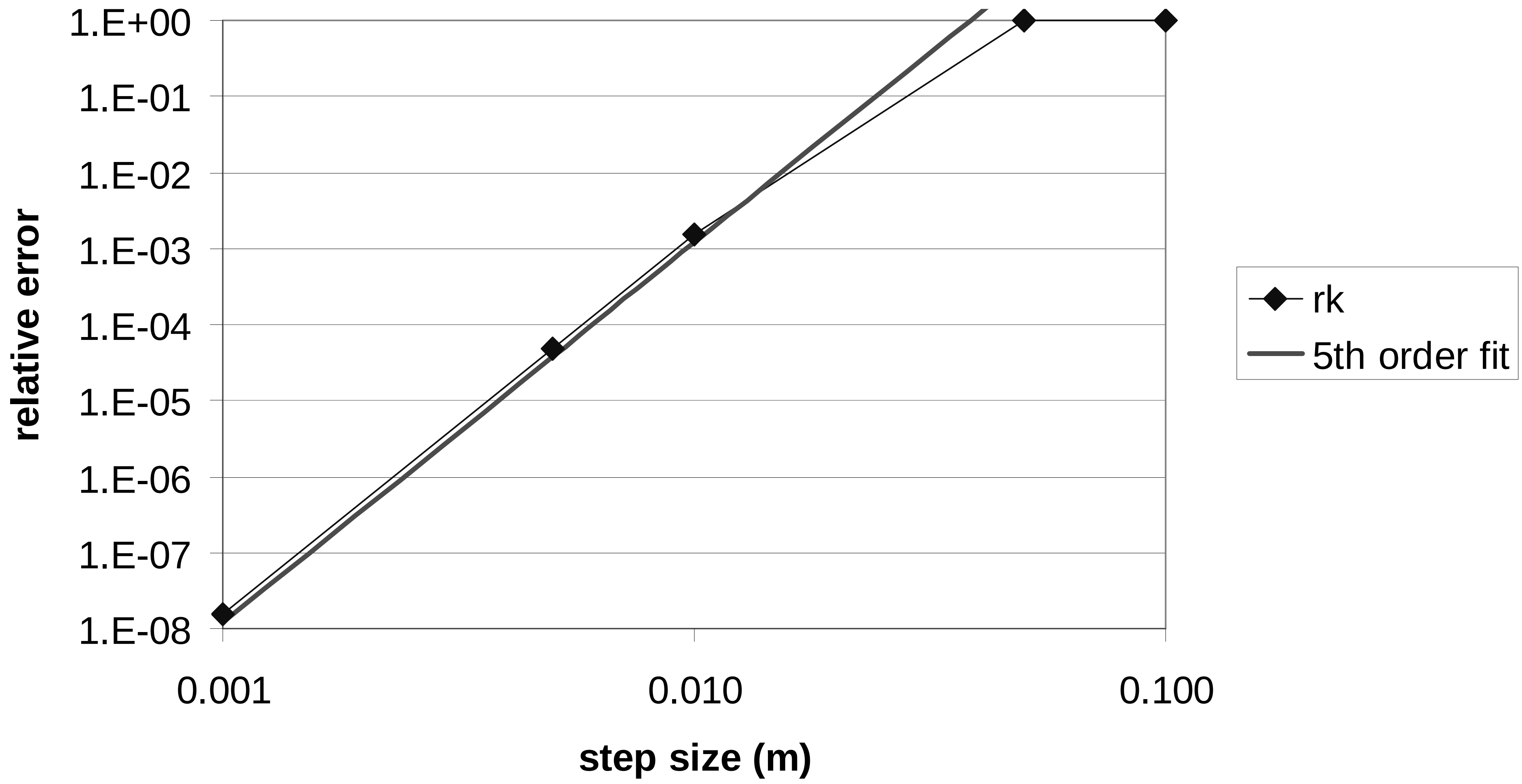
FOFO: relative error in angular momentum (log-log)



Runge Kutta errors

Muon Collaboration

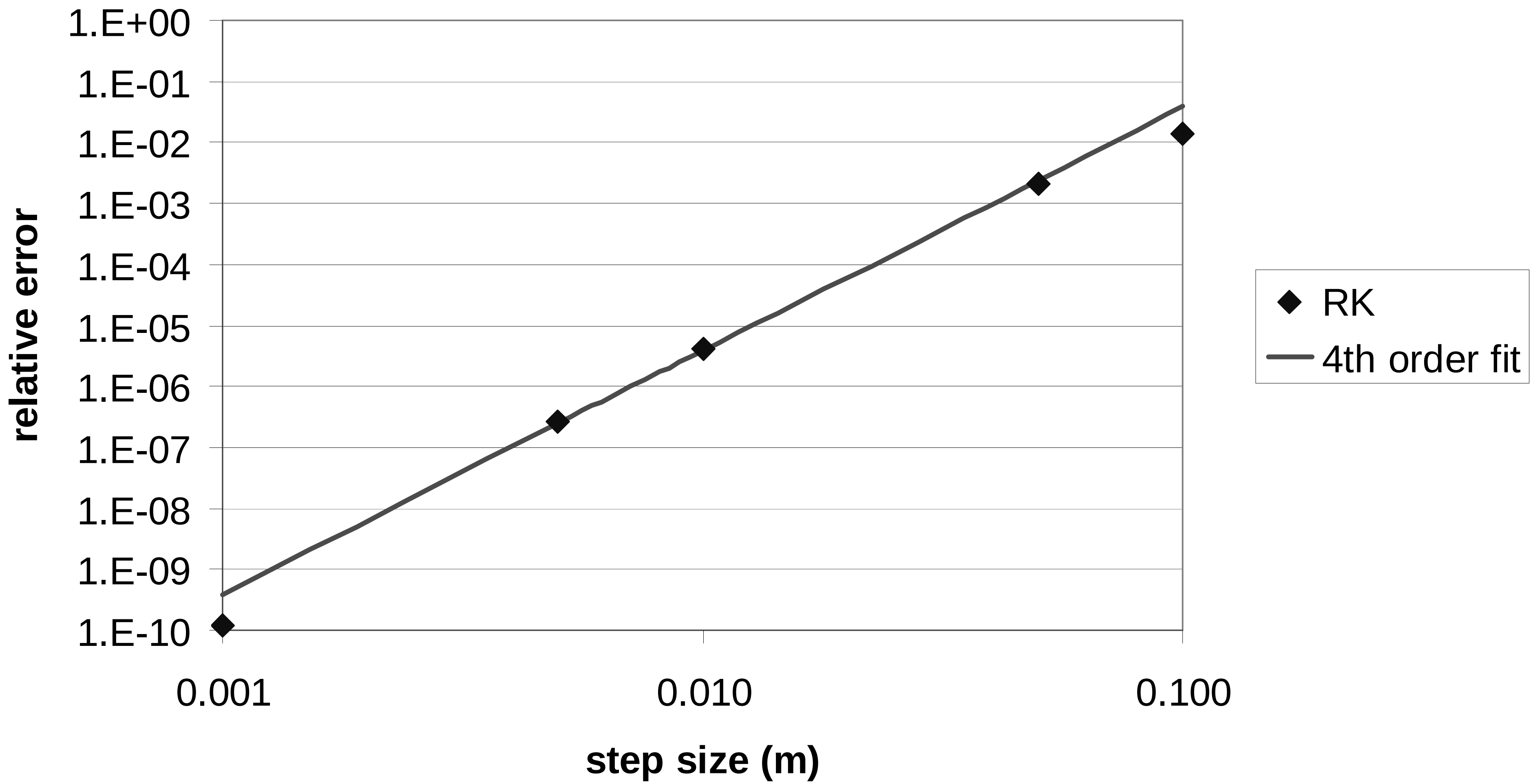
Solenoid: relative error in P⊥ (log-log)



Runge Kutta Errors

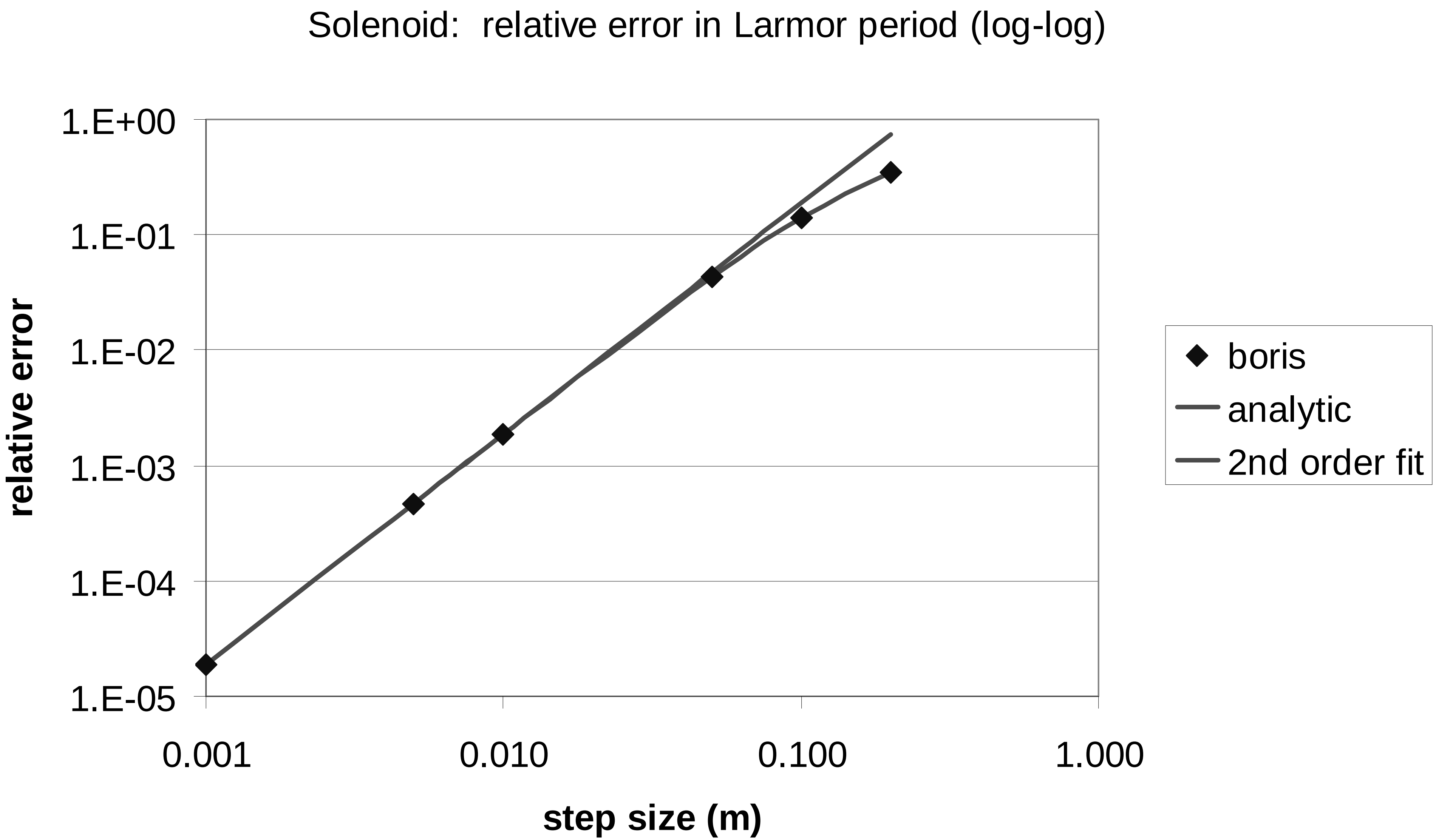
Muon Collaboration

Solenoid: relative error in Larmor period (log-log)



Boris errors in Solenoid

Muon Collaboration



References

Muon Collaboration

P. H. Stoltz, G. Penn, J. Wurtele, and J. R. Cary, MC Note 229.

J. Boris, *Proc. of the 4th Conf. on Numerical Simulation of Plasma* (NRL, 1970).

A. Dullweber, B. Leimkuhler, and R. McLachlan, J. Chem. Phys. **15** (1997) 5840**.**