Normalized emittance in the case of large momentum spreads

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Abstract. The standard definition of the normalized emittance uses an average $\beta\gamma$ to correct for the acceleration. In the case of large momentum spreads, especially in combination with an asymmetric momentum distribution (as it is the case in the Neutrino Factory or related cooling experiments), this definition yields to a systematic over- or underestimation of the normalized emittance. An improved way for calculating the normalized emittance is proposed here.

1. MICE

Ionization cooling plays a major role in most existing designs for a Neutrino Factory. As it has never been tested before, an international Muon Ionization Cooling Experiment (MICE)[1] has been proposed. It consists of three main parts: the input diagnostics, the cooling device itself and the output diagnostics. The idea is to compare the emittance measurement before and after the cooling apparatus to proove cooling. This emittance measurement is special in some ways. MICE plans to use the *single particle method* and for the first time, emittance is measured following the statistical definition by tracking each individual particle in the beam instead of observing cumulative quantities. The emittance is subsequently calculated in much the same way as in particle tracking programs such as PATH[2]. It is anticipated to be much more precise than any emittance measurement performed so far.

Up to now, great emphasis has been laid on the resolution of the tracking detector. The emittance calculation, on the other hand, has attracted less attention. This paper argues, that there are some unresolved detail issues in the definition of the normalized emittance that have not mattered up to this date, but will matter in a cooling experiment. The conclusions should be valid for similar cooling experiments that have been proposed so far.

2. Cooling, Emittance definition

Cooling can be defined as the reduction of the normalized emittance ε_N without particle loss (emittance reduction point of view). This is directly related to an increase in the phase space density at the center of the beam, which is an alternative definition of cooling (particle density point of view).

The current definition[3] of the *rms* ε_N is to first calculate the geometric emittance using P. Lapostolle's definition[4] and to normalize afterwards using an average $\beta\gamma$. In the two dimensional case, this would be:

$$\varepsilon_N = \overline{\beta\gamma}\varepsilon = \langle\beta_i\gamma_i\rangle^2\Big(\langle x_i^2\rangle\langle x_i'^2\rangle - \langle x_i x_i'\rangle^2\Big) \tag{1}$$

This leads to imprecisions when there is a nominal energy spread, because all particles are treated as if they had the same momentum:

- In the *emittance reduction* point of view, the contribution of low-energy particles (that have a low $\beta\gamma$), is multiplied by the higher $\overline{\beta\gamma}$. This leads to overestimating the contribution of low-energy particles while underestimating the contribution of high-energy particles to the overall emittance.
- In the *particle density* point of view, low-energy particles, again, are normalized with the higher $\overline{\beta\gamma}$. This leads to rejecting low-energy particles with higher x' that still would fit into the acceptance while accepting high-energy particles with moderate x' that in reality would already hit the beam pipe.

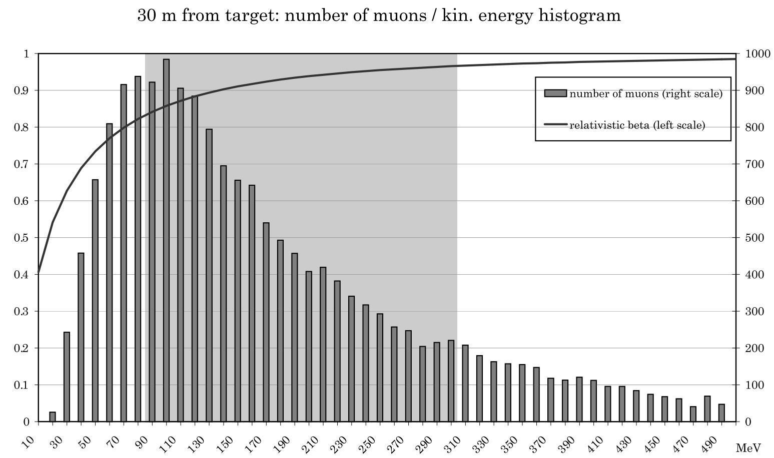

Figure 1. Energy distribution immediately after phase rotation in the CERN neutrino factory study. The momentum range of the useful muons is marked grey. As there are more low-energy particles, the normalized emittance is over-estimated. From [5].

Both phenomena are negligible in the case of small energy spreads or a symmetric momentum distribution. Due to the pion production and capture process in MICE as well as in the Neutrino Factory, this is *not* the case. In the CERN Neutrino Factory[5], as it can be seen from Fig 2, the momentum distribution immediately after the phase rotation is shifted towards lower momenta. This leads to a systematic overestimation of the normalized emittance in all stages of the cooling channel.

In classical machines, this effect has never needed consideration, because $\Delta p/p$ is always much less than $\pm 1\%$. In MICE as well as in the Neutrino Factory, this value will be in the order of magnitude of $\pm$10-20%, requiring a more precise definition of the normalized emittance.

3. Proposed correction

A solution can be a modified normalization process. Instead of using Eq. 1, it would be more correct to normalize each particle i with its own $\beta_i\gamma_i$. In the two dimensional case this would be:

$$\varepsilon^2_{N,new} = \langle x_i^2 \rangle \langle \beta_i^2 \gamma_i^2 x_i'^2 \rangle - \langle x_i x_i' \beta_i \gamma_i \rangle^2 \tag{2}$$

It is important to note that the $\beta_i\gamma_i$ term has been added to the x_i' term, as it originates from the fact that x' is not a canonical variable. In fact, the above equation can be traced back to a definition of the phase space. Defining the 2D-phase space f as:

$$f^2 = \langle x_i^2 \rangle \langle p_{x,i}^2 \rangle - \langle x_i p_{x,i} \rangle^2 \tag{3}$$

and using the identity

$$p_x = \frac{p_x}{p} \frac{p}{m_0 c} m_0 c = x' \beta \gamma m_0 c \tag{4}$$

it can be shown that the new definition of the normalized emittance is the phase space volume f divided by m_0c:

$$f^2 = m_0c\Big(\langle x_i^2\rangle\langle\beta_i^2\gamma_i^2x_i'^2\rangle - \langle x_ix_i'\beta_i\gamma_i\rangle^2\Big) \tag{5}$$

A first estimate shows, that the 2D normalized emittance has been overestimated by 10% in the case of the CERN Neutrino Factory.

4. Conclusion

While the correction described in this paper is a second order effect and can be neglected in many cases, it changes the emittance calculation notably for the Neutrino Factory and related cooling experiments. This is especially the case for MICE, where the aim is to measure the emittance with a resolution of a few permille. A code for calculating the normalized emittance in the here proposed way is under development and may be used in MICE.

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