Analytic Formulation of Neutrino Oscillation Probability in Constant Matter

Keiichi Kimura† §, Akira Takamura†‡ and Hidekazu Yokomakura†

† Department of Physics, Nagoya University, *Nagoya, 464-8602, Japan*
‡ Department of Mathematics, Toyota National Collage of Technology, *Eisei-cho 2-1, Toyota-shi, 471-8525, Japan*

Abstract. In this talk, based on the work [1], we present the simple derivation of exact and analytic formula for neutrino oscillation probability. We consider three flavor neutrino oscillations in matter with constant density.

The information on the neutrino masses and the leptonic mixing angles has been accumulated gradually in recent neutrino experiments. In this situation, Superbeam [2] and Neutrino factory [3] experiments using artificial neutrino beam are planned in near future to determine the unknown parameters. In particular, the measurement of the leptonic CP phase δ is the final goal of the neutrino experiments and many researches have been done [4]. We here focus on the analytic formulation of neutrino oscillation probability in matter with constant density and the CP dependence of the probability.

We take the channel for $\nu_e \to \nu_\mu$ transition, which is considered to be golden channel for the measurement of CP violation. In vacuum, the oscillation probability is given by

$$P(\nu_e \to \nu_\mu) = -4 \sum_{(ij)}^{\text{cyclic}} \mathrm{Re} J_{e\mu}^{ij} \sin^2 \Delta'_{ij} - 2 \sum_{(ij)}^{\text{cyclic}} J \sin 2\Delta'_{ij}, \tag{1}$$

where

$$J_{e\mu}^{ij} \equiv U_{ei} U_{\mu i}^* (U_{ej} U_{\mu j}^*)^*, \quad J \equiv \mathrm{Im}\, J_{e\mu}^{12}, \quad \Delta'_{ij} \equiv \frac{L}{4E} \Delta_{ij}, \tag{2}$$

and U is the Maki-Nakagawa-Sakata (MNS) matrix [5]. In matter, the Hamiltonian changes as

$$\tilde{H} = H + \frac{1}{2E} \mathrm{diag}(a, 0, 0), \tag{3}$$

where a is the matter potential. According to this change, the mass eigenvalues and the MNS matrix also change to the effective ones. Their concrete expressions are calculated in [6, 7, 8]. However, it is difficult to simplify these expressions and understand the CP dependence of the probability exactly.

We here use the method by Naumov [9], Harrison and Scott [10]. In the method, they notice the invariance of the Hamiltonian except for H_{ee}. They consider the product of off-diagonal elements of Hamiltonian as

$$\mathrm{Im}(\tilde{H}_{e\mu} \tilde{H}_{\mu\tau} \tilde{H}_{\tau e}) = \mathrm{Im}(H_{e\mu} H_{\mu\tau} H_{\tau e}), \tag{4}$$

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and rewrite the Hamiltonian by using the relation

$$\tilde{H} = \tilde{U}\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\tilde{U}^\dagger. \tag{5}$$

As a result, they obtain a simple and elegant identity,

$$\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}\tilde{J} = \Delta_{12}\Delta_{23}\Delta_{31}J. \tag{6}$$

¿From this identity, one can see that the matter effect is only included in the effective masses and CP phase is included in J. Thus, $\tilde{J}$, which is the product of four $\tilde{U}_{\alpha i}$s, has very simple form. This suggests that the matter effect and the CP phase included in a $\tilde{U}_{\alpha i}$ partially cancel with those in other $\tilde{U}_{\beta j}$. We show that $\mathrm{Re}\tilde{J}^{ij}_{e\mu}$ is also calculated as in the same way.

Here, remember the expression for the oscillation probability in (1)-(2). One can see that $\tilde{U}_{ei}$ always appears with $\tilde{U}^*_{\mu i}$. Namely, we only have to calculate $\tilde{U}_{ei}\tilde{U}^*_{\mu i}$ not $\tilde{U}_{\alpha i}$ to obtain the oscillation probability. We construct three identities on $\tilde{U}_{ei}\tilde{U}^*_{\mu i}$ by using the invariance of the Hamiltonian. First one is obtained from the unitarity as

$$\tilde{U}_{e1}\tilde{U}^*_{\mu 1} + \tilde{U}_{e2}\tilde{U}^*_{\mu 2} + \tilde{U}_{e3}\tilde{U}^*_{\mu 3} = 0. \tag{7}$$

Second one is obtained from $\tilde{H}_{e\mu} = H_{e\mu}$. The Hamiltonian is rewritten by the MNS matrix from (5) as in the previous case and we obtain

$$\lambda_1\tilde{U}_{e1}\tilde{U}^*_{\mu 1} + \lambda_2\tilde{U}_{e2}\tilde{U}^*_{\mu 2} + \lambda_3\tilde{U}_{e3}\tilde{U}^*_{\mu 3} = p, \tag{8}$$

where $p \equiv H_{e\mu}/(2E)$. Third one is obtained from $\tilde{H}_{\tau\mu}\tilde{H}_{e\tau} - \tilde{H}_{e\mu}\tilde{H}_{\tau\tau} = H_{\tau\mu}H_{e\tau} - H_{e\mu}H_{\tau\tau}$ as

$$\lambda_2\lambda_3\tilde{U}_{e1}\tilde{U}^*_{\mu 1} + \lambda_3\lambda_1\tilde{U}_{e2}\tilde{U}^*_{\mu 2} + \lambda_1\lambda_2\tilde{U}_{e3}\tilde{U}^*_{\mu 3} = q, \tag{9}$$

where $q \equiv (H_{\tau\mu}H_{e\tau} - H_{e\mu}H_{\tau\tau})/(2E)^2$. It is noted that these are independent of the matter effects. Regarding these three identities as the equations on $\tilde{U}_{ei}\tilde{U}^*_{\mu i}$, we simply obtain the expression ‖ as

$$\tilde{U}_{ei}\tilde{U}^*_{\mu i} = \frac{p\lambda_i + q}{\tilde{\Delta}_{ji}\tilde{\Delta}_{ki}}. \tag{10}$$

Note that the matter effect is included only in the effective masses as in the case of $\tilde{J}$. How about the CP phase ? In the standard parametrization

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{11}$$

the CP phase is not included in the expressions for λ_i at all [6]. On the other hand, p and q are given by

$$p = \Delta_{21}U_{e2}U^*_{\mu 2} + \Delta_{31}U_{e3}U^*_{\mu 3}, \quad q = \Delta_{21}\Delta_{31}U_{e1}U^*_{\mu 1}. \tag{12}$$

Substituting (12) into (10), we find that the oscillation probability has the form

$$P(\nu_e \to \nu_\mu) = A\cos\delta + B\sin\delta + C \tag{13}$$

with respect to δ [1]. Note that the probability has only linear terms of $\cos\delta$ and $\sin\delta$ exactly¶. This is one of our main results. We can further calculate A, B and C as

$$A = \sum_{(ijk)}^{\text{cyclic}} \frac{-8J_r\Delta_{21}[\Delta_{31}\lambda_k(\lambda_k - \Delta_{31}) + A_k]}{\tilde{\Delta}^2_{jk}\tilde{\Delta}^2_{ki}} \cos\tilde{\Delta}'_{ij}\sin\tilde{\Delta}'_{jk}\sin\tilde{\Delta}'_{ki}, \tag{14}$$

‖ Similar expression is also seen in Ref. [11, 12]

¶ It is shown in Ref. [13] that the CP-odd part is linear in $\sin\delta$.

$$B = \frac{8\Delta_{12}\Delta_{23}\Delta_{31}}{\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} J_r \sin\tilde{\Delta}'_{12} \sin\tilde{\Delta}'_{23} \sin\tilde{\Delta}'_{31}, \tag{15}$$

$$C = \sum_{(ij)}^{\text{cyclic}} \frac{-4[s_{13}^2 s_{23}^2 c_{13}^2 \Delta_{31}^2 \lambda_i \lambda_j + C_{ij}]}{\tilde{\Delta}_{ij}\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} \sin^2\tilde{\Delta}'_{ij}, \tag{16}$$

where A_k and C_{ij} include the higher terms of Δ_{21} and can be neglected in the limit $\Delta_{21} \ll \Delta_{31}$. See [1] for their concrete expressions.

As one of the merit of our formula, one can simply reproduce the well-known approximate formulas given in [14, 15] from (13)-(16). Another merit is that we can evaluate δ directly in terms of the probabilities. The probability of anti-neutrino has also the same form as that of neutrino

$$P(\bar{\nu}_e \to \bar{\nu}_\mu) = \bar{A}\cos\delta + \bar{B}\sin\delta + \bar{C}. \tag{17}$$

¿From (13) and (17), we obtain the analytic expressions for δ as

$$\sin\delta = \frac{\{\bar{A}P(\nu_e \to \nu_\mu) - AP(\bar{\nu}_e \to \bar{\nu}_\mu)\} - (\bar{A}C - A\bar{C})}{\bar{A}B - A\bar{B}}, \tag{18}$$

$$\cos\delta = \frac{\{\bar{B}P(\nu_e \to \nu_\mu) - BP(\bar{\nu}_e \to \bar{\nu}_\mu)\} - (\bar{B}C - B\bar{C})}{\bar{B}A - B\bar{A}}. \tag{19}$$

If the probabilities of neutrino and anti-neutrino are precisely determined by future experiments, we can calculate the value of δ analytically by using (18)-(19)[+]. Thus, we expect that our formula is useful to understand the features of neutrino physics and to obtain the information on the leptonic CP phase.

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[+] It is pointed out that there are ambiguities for the measurement of CP phase δ due to the eight-fold degeneracy of other parameters [16, 17, 18, 19]. However, we do not discuss the problem here.