Exact Formula of Probability and CP Violation for Neutrino Oscillations in Matter

3 July 2002 Keiichi Kimura
Nagoya University
hep-ph/0203099

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Collabolated with
Akira Takamura
Hidekazu Yokomakura

Introduction

Present

- Atmospheric Neutrino

$$\theta_{23} \sim \frac{\pi}{4}, \qquad |\Delta_{31}| \sim 3 \times 10^{-3}\mathrm{eV}^2$$

- Solar Neutrino

$$\theta_{12} \sim \frac{\pi}{4}, \qquad |\Delta_{21}| \sim 1 \times 10^{-4}\mathrm{eV}^2$$

- CHOOZ

$$\theta_{13} \leq 0.2$$

$$J = \underbrace{s_{12}c_{12}}_{0.5}\underbrace{s_{23}c_{23}}_{0.5}s_{13}c_{13}^2 \sin\delta$$

Future

- Superbeam or Neutrino Factory

$$\theta_{13}, \qquad \mathrm{sgn}(\Delta_{31}), \qquad \delta$$

Our Main Purpose

Measurement of the CP phase

Neutrino Oscillation Probability

$\nu_e \to \nu_\mu$ transition

$$P(\nu_e \to \nu_\mu) = \left| < \nu_\mu | U e^{-iHt} U^\dagger | \nu_e > \right|^2$$

$U_{\alpha i}$: MNS matrix

$$U^\dagger H U = \frac{1}{2E}\mathrm{diag}(0, \Delta_{21}, \Delta_{31})$$

$$P(\nu_e \to \nu_\mu) = \underbrace{-4 \sum_{(ij)}^{\text{cyclic}} \mathrm{Re} J^{ij}_{e\mu} \sin^2 \Delta'_{ij}}_{\text{CP even}} \underbrace{-2 \sum_{(ij)}^{\text{cyclic}} J \sin 2\Delta'_{ij}}_{\text{CP odd}}$$

$$J^{ij}_{e\mu} \equiv U_{ei} U^*_{\mu i} (U_{ej} U^*_{\mu j})^* \quad J \equiv \mathrm{Im}\, J^{12}_{e\mu} \quad \Delta'_{ij} \equiv \frac{L}{4E}\Delta_{ij}$$

vacuum $\Longrightarrow$ matter

$$H \Longrightarrow \tilde{H} = H + \frac{1}{2E} \begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad \tilde{U}^\dagger \tilde{H} \tilde{U} = \frac{1}{2E}\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$$

Probability in matter is obtained by the replacements

$$\begin{cases} U_{\alpha i} \to \tilde{U}_{\alpha i} \\ \Delta_{ij} \to \tilde{\Delta}_{ij} \equiv \lambda_i - \lambda_j \end{cases}$$

Approach to CP phase and Problem

Vacuum

$$\begin{aligned} \Delta P_{CP} &= P(\nu_e \to \nu_\mu; \delta) - P(\bar{\nu}_e \to \bar{\nu}_\mu; -\delta) \\ &= -4 \sum_{(ij)}^{\text{cyclic}} J \sin 2\Delta'_{ij} \propto \sin\delta \end{aligned}$$

$\Delta P_{CP} \neq 0 \Rightarrow$ CP violation (Vacuum)

Matter

$$\begin{aligned} \Delta P_{CP} &= P(\nu_e \to \nu_\mu; \delta, a) - P(\bar{\nu}_e \to \bar{\nu}_\mu; -\delta, -a) \\ &\neq -4 \sum_{(ij)}^{\text{cyclic}} \tilde{J} \sin 2\tilde{\Delta}'_{ij} \propto \sin\delta \end{aligned}$$

$\Delta P_{CP} \neq 0 \Rightarrow$ CP violation or Matter Effects (Matter)

$\star$ We focus on the exact understanding of pure CP effects and Matter effects in $P(\nu_e \to \nu_\mu)$.

Assumptions

- Constant Matter
- Standard Three Active Neutrino Scenario.

Direct Approach (Effective MNS matrix)

$$\tilde{U}_{\alpha i} = \frac{N_i}{D_i} U_{\alpha i} + \frac{a}{D_i} U_{ei} \left[\left(\lambda_i - m_j^2 \right) U_{ek}^* U_{\alpha k} + \left(\lambda_i - m_k^2 \right) U_{ej}^* U_{\alpha j} \right]$$

$$\begin{cases} N_i = \left(\lambda_i - m_j^2 \right) \left(\lambda_i - m_k^2 \right) - a \left[\left(\lambda_i - m_j^2 \right) |U_{ek}|^2 + \left(\lambda_i - m_k^2 \right) |U_{ej}|^2 \right] \\ D_i^2 = N_i^2 + a^2 |U_{ei}|^2 \left[\left(\lambda_i - m_j^2 \right)^2 |U_{ek}|^2 + \left(\lambda_i - m_k^2 \right)^2 |U_{ej}|^2 \right] \end{cases}$$

Z. Xing, hep-ph/0002246

$$\tilde{J}_{e\mu}^{ij} = \tilde{U}_{ei} \tilde{U}_{\mu i}^* (\tilde{U}_{ej} \tilde{U}_{\mu j}^*)^* = ?$$

Both Matter effects and CP effects are complicated.

Matter Invariant Approach

$$\tilde{H} = H + \frac{1}{2E}\begin{pmatrix} a & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$\Longrightarrow \tilde{H}_{\alpha\beta} = H_{\alpha\beta} \quad (\text{except for } H_{ee})$$

We can make matter invariavt identities by using this relation.

$$\mathrm{Im}(\tilde{H}_{e\mu}\tilde{H}_{\mu\tau}\tilde{H}_{\tau e}) = \mathrm{Im}(H_{e\mu}H_{\mu\tau}H_{\tau e})$$

$$\begin{aligned} \tilde{H}_{\alpha\beta} &= \lambda_1\tilde{U}_{\alpha 1}\tilde{U}^*_{\beta 2} + \lambda_2\tilde{U}_{\alpha 2}\tilde{U}^*_{\beta 2} + \lambda_3\tilde{U}_{\alpha 3}\tilde{U}^*_{\beta 3} \\ (\tilde{H} &= \tilde{U}\,\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)\tilde{U}^\dagger) \end{aligned}$$

$$\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}\tilde{J} = \Delta_{12}\Delta_{23}\Delta_{31}J$$

(Naumov-Harrison-Scott identity)

$$\Longrightarrow \tilde{J} = \mathrm{Im}(\tilde{U}\tilde{U}^*\tilde{U}^*\tilde{U}) = \frac{\Delta_{12}\Delta_{23}\Delta_{31}}{\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}}J$$

$$J = s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin\delta$$

Matter Effects $\Longrightarrow$ $\tilde{\Delta}_{ij}$
CP Effects $\Longrightarrow$ $\sin\delta$

Both matter effects and CP effects are simple.
Can we also derive the expression of $\mathrm{Re}\tilde{J}$?

Simple Derivation of Exact Formula

$$P(\nu_e \to \nu_\mu) = -4 \sum_{(ij)}^{\text{cyclic}} \text{Re}\tilde{J}^{ij}_{e\mu} \sin^2 \tilde{\Delta}'_{ij} - 2 \sum_{(ij)}^{\text{cyclic}} \tilde{J} \sin 2\tilde{\Delta}'_{ij}$$

$$\tilde{J}^{ij}_{e\mu} = \tilde{U}_{ei}\tilde{U}^*_{\mu i}(\tilde{U}_{ej}\tilde{U}^*_{\mu j})^* \quad \tilde{J} = \text{Im}\tilde{J}^{12}_{e\mu}$$

We only have to calculate $\tilde{U}_{ei}\tilde{U}^*_{\mu i}$ to learn $P(\nu_e \to \nu_\mu)$.

Unitarity

$$\tilde{U}_{e1}\tilde{U}^*_{\mu 1} + \tilde{U}_{e2}\tilde{U}^*_{\mu 2} + \tilde{U}_{e3}\tilde{U}^*_{\mu 3} = 0$$

$\tilde{H}_{e\mu} = H_{e\mu}$

$$\lambda_1\tilde{U}_{e1}\tilde{U}^*_{\mu 1} + \lambda_2\tilde{U}_{e2}\tilde{U}^*_{\mu 2} + \lambda_3\tilde{U}_{e3}\tilde{U}^*_{\mu 3} = p$$

$\tilde{\mathcal{H}}_{e\mu} = \mathcal{H}_{e\mu}$

$$\lambda_2\lambda_3\tilde{U}_{e1}\tilde{U}^*_{\mu 1} + \lambda_3\lambda_1\tilde{U}_{e2}\tilde{U}^*_{\mu 2} + \lambda_1\lambda_2\tilde{U}_{e3}\tilde{U}^*_{\mu 3} = q$$

$$(\mathcal{H}_{e\mu} \equiv H_{e\tau}H_{\tau\mu} - H_{e\mu}H_{\tau\tau})$$

$$\Rightarrow \tilde{U}_{ei}\tilde{U}^*_{\mu i} = \frac{p\lambda_i + q}{\tilde{\Delta}_{ji}\tilde{\Delta}_{ki}} \qquad \begin{cases} p = 2EH_{e\mu} \\ q = (2E)^2\mathcal{H}^*_{e\mu} \end{cases}$$

Matter Effect $\Longrightarrow$ λ_i

CP Effects $\Longrightarrow$ λ_i, p, q?

Effective Masses

$$
\begin{aligned}
\lambda_1 &= \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t}\left[u + \sqrt{3(1-u^2)}\right] \\
\lambda_2 &= \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t}\left[u - \sqrt{3(1-u^2)}\right] \\
\lambda_3 &= \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2 - 3t}
\end{aligned}
$$

$$
\begin{aligned}
s &= \Delta_{21} + \Delta_{31} + a \\
t &= \Delta_{21}\Delta_{31} + a[\Delta_{21}(1 - s_{12}^2 c_{13}^2) + \Delta_{31}(1 - s_{13}^2)] \\
u &= \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2s^3 - 9st + 27a\Delta_{21}\Delta_{31}c_{12}^2 c_{13}^2}{2(s^2 - 3t)^{3/2}}\right)\right]
\end{aligned}
$$

Remark

Independent of CP phase δ

Matter Invariant Quantities p and q

$$\begin{cases} p = \Delta_{21} U_{e2} U^*_{\mu 2} + \Delta_{31} U_{e3} U^*_{\mu 3} \\ q = \Delta_{21}\Delta_{31} U_{e1} U^*_{\mu 1} \end{cases}$$

⋆ Standard Parametrization

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$$p = p_1 e^{-i\delta} + p_2, \quad q = q_1 e^{-i\delta} + q_2$$

$$\Downarrow \qquad \tilde{U}_{ei}\tilde{U}^*_{\mu i} = \frac{p\lambda_i + q}{\tilde{\Delta}_{ji}\tilde{\Delta}_{ki}}$$

$$\boxed{\tilde{U}_{ei}\tilde{U}^*_{\mu i} = X_1 e^{-i\delta} + X_2}$$

CP dependence of $P(\nu_e \to \nu_\mu)$

$$P(\nu_e \to \nu_\mu) = A\cos\delta + B\sin\delta + C$$

$P(\nu_e \to \nu_\mu)$ has only linear terms in $\cos\delta$ and $\sin\delta$.

Exact

$$A = \sum_{(ijk)}^{\text{cyclic}} \frac{-8J_r\Delta_{21}[\Delta_{31}\lambda_k(\lambda_k-\Delta_{31})+A_k]}{\tilde{\Delta}^2_{jk}\tilde{\Delta}^2_{ki}} \cos\tilde{\Delta}'_{ij}\sin\tilde{\Delta}'_{jk}\sin\tilde{\Delta}'_{ki}$$

$$B = \frac{8\Delta_{12}\Delta_{23}\Delta_{31}}{\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} J_r \sin\tilde{\Delta}'_{12}\sin\tilde{\Delta}'_{23}\sin\tilde{\Delta}'_{31}$$

$$C = \sum_{(ij)}^{\text{cyclic}} \frac{-4[s^2_{13}s^2_{23}c^2_{13}\Delta^2_{31}\lambda_i\lambda_j + C_{ij}]}{\tilde{\Delta}_{ij}\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} \sin^2\tilde{\Delta}'_{ij}$$

$$\lambda_1 \sim \Delta_{21},\ \lambda_2 \sim \Delta_{31},\ \lambda_3 \sim a$$
$$(\lambda_1 \sim \Delta_{21},\ \lambda_2 \sim a,\ \lambda_3 \sim \Delta_{31})$$

Approximate

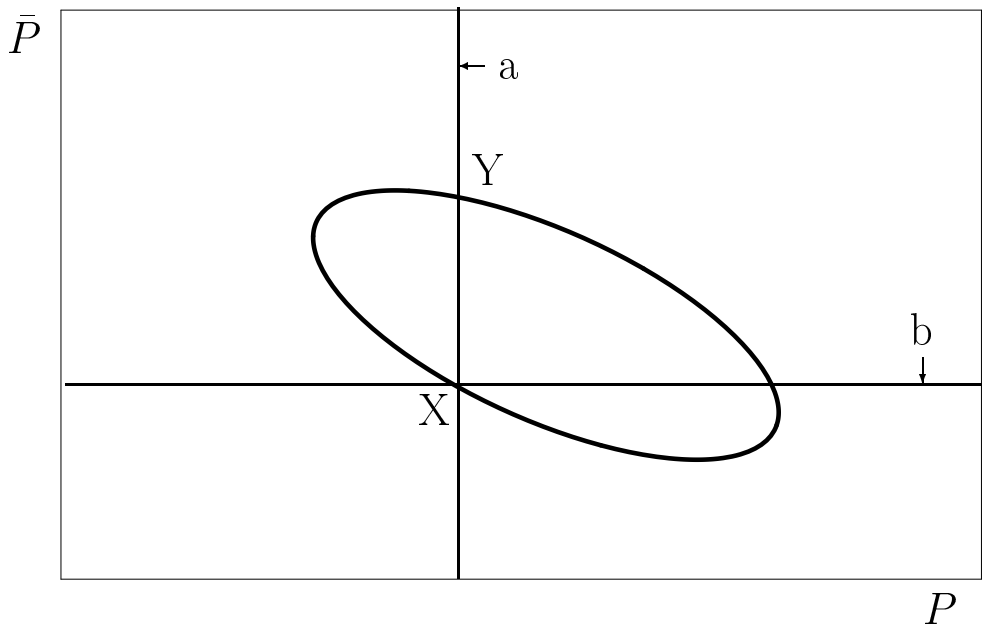
$$A \simeq \frac{8J_r\Delta_{21}\Delta_{31}}{a(\Delta_{31}-a)} \cos\Delta'_{31}\sin a'\sin(\Delta_{31}-a)'$$

$$B \simeq \frac{8J_r\Delta_{21}\Delta_{31}}{a(\Delta_{31}-a)} \sin\Delta'_{31}\sin a'\sin(\Delta_{31}-a)'$$

$$C \simeq \frac{4\Delta^2_{31}s^2_{23}s^2_{13}c^2_{13}}{(\Delta_{31}-a)^2} \sin^2(\Delta_{31}-a)'$$

Expression for CP phase

CP trajectory



H. Minakata and H. Nunokawa, hep-ph/0108085

$$P(\nu_e \to \nu_\mu) = A\cos\delta + B\sin\delta + C$$
$$\bar{P}(\bar{\nu}_e \to \bar{\nu}_\mu) = \bar{A}\cos\delta + \bar{B}\sin\delta + \bar{C}$$

CP phase can be determined by

$$\sin\delta = \frac{(\bar{A}P - A\bar{P}) - (\bar{A}C - A\bar{C})}{\bar{A}B - A\bar{B}}$$
$$\cos\delta = \frac{(\bar{B}P - B\bar{P}) - (\bar{B}C - B\bar{C})}{\bar{B}A - B\bar{A}}$$

Conclusion

⋆ We have derived an exact and simple formula of Neutrino Oscillation Probability.

Three Identities $\Rightarrow \tilde{U}_{ei}\tilde{U}^*_{\mu i} \Rightarrow \tilde{J}^{ij}_{e\mu} \Rightarrow P(\nu_e \to \nu_\mu)$

⋆ We have found the CP dependence of $P(\nu_e \to \nu_\mu)$.

$$P(\nu_e \to \nu_\mu) = A\cos\delta + B\sin\delta + C$$