Nuclear and Particle Physics - Lecture 2
Relativistic kinematics

1 Introduction

In particle physics, we often deal with particles travelling close to the speed of light; photons of course always go at the speed of light! Hence, we need to review the formulæ for relativistic kinematics.

2 Energy and momentum

The usual notation in relativity is that the velocity \( v \) can be used to define the dimensionless quantities

\[
\beta = \frac{v}{c}, \quad |\beta| < 1
\]

and

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \gamma \geq 1
\]

For a particle with a non-zero mass, the total energy, including the energy due to the mass, is given by

\[
E = \gamma mc^2
\]

Note, at rest, \( \beta = 0 \) so \( \gamma = 1 \) and hence \( E = mc^2 \), the famous Einstein equation. The momentum is

\[
p = \gamma mv = \gamma mc\beta
\]

These can easily be inverted to give \( \beta \) and \( \gamma \) in terms of \( p \) and \( E \). Clearly

\[
\gamma = \frac{E}{mc^2} \geq 1
\]

so that the energy is always greater than or equal to the mass times \( c^2 \). Similarly

\[
\beta = \frac{pc}{E}
\]

and since \( |\beta| < 1 \), then

\[
\frac{|p|c}{E} < 1
\]

i.e. the momentum magnitude is always less than the energy. There is another important relationship which results from the above; consider

\[
E^2 - p^2 c^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 c^4 \beta^2 = \gamma^2 m^2 c^4 (1 - \beta^2) = m^2 c^4
\]

or

\[
E^2 = p^2 c^2 + m^2 c^4
\]

We will use this equation very often in the course. Note, \( E \) here is the total energy; the kinetic energy \( T \) is (by definition) the extra energy due to the motion and so is

\[
T = E - mc^2 = (\gamma - 1)mc^2
\]
To connect this with the classical equations for $T$ and $p$ which you are familiar with, then we need to consider the non-relativistic limit, where $|v| \ll c$, or equivalently $|\beta| \ll 1$. In this limit

$$\gamma = (1 - \beta^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-\beta^2) = 1 + \frac{\beta^2}{2}$$

Hence

$$E = \gamma mc^2 \approx mc^2 + \frac{1}{2}mc^2\beta^2 = mc^2 + \frac{1}{2}mv^2$$

and so

$$T = E - mc^2 \approx \frac{1}{2}mv^2$$

as expected. For $p$, we only need to consider the first term in $\gamma$, i.e. $\gamma \approx 1$, so

$$p = \gamma mv \approx mv$$

Finally, for massless particles such as the photon, the above equations do not work as $|\beta| = 1$ and hence $\gamma$ goes infinite. The formula to use is the one where $\gamma$ does not appear, namely

$$E^2 = p^2c^2 + m^2c^4 \to p^2c^2$$

for the case of $m = 0$, which then simply means

$$E = |p|c$$

Obviously, there is no non-relativistic approximation for this case.

There is one more very important fundamental property of $E$ and $pc$ which is that under a Lorentz transformation, they transform in the same way as $ct$ and $r$. (Both of these combinations form what is called “four-vectors”, which most of you will have met in the Advanced Classical Physics course.) Specifically, for a boost by $v_b$ along the $z$ direction, then the boosted values are

$$E' = \gamma_b \left(E - \frac{v_b}{c}p_zc\right), \quad p'_z = \gamma_b \left(p_zc - \frac{v_b}{c}E\right)$$

while $p'_x = p_x$ and $p'_y = p_y$. Now consider

$$m^2c^4 = E'^2 - p'^2c^2$$

$$= \gamma_b^2 \left(E^2 - 2Ev_bp_z + v_b^2p_z^2\right) - p_x'^2c^2 - p_y'^2c^2 - \gamma_b^2 \left(p_x^2c^2 - 2Evbpz + \frac{v_b^2E^2}{c^2}\right)$$

$$= \gamma_b^2 \left(E^2 - p_z^2c^2\right) \left(1 - \frac{v_b^2}{c^2}\right) - p_x'^2c^2 - p_y'^2c^2$$

$$= E^2 - p_x'^2c^2 - p_y'^2c^2 - p_z'^2c^2 = m^2c^4$$

Hence, the mass is invariant, meaning it does not depend on the Lorentz frame. This is the “correct” way to consider mass; concepts like “rest mass” etc. are not very useful.

Note, since energy is often expressed in units of (e.g.) MeV, then, as momentum has units of $E/c$ then it is often expressed in units of MeV/c. Similarly, mass is expressed in MeV/c$^2$. 

2
3 Multiparticle systems

In a collision or decay, there will be more than one particle involved. The total energy $E_T = \sum_i E_i$ and total momentum $p_T = \sum_i p_i$ are always conserved; this is a very fundamental principle. Although their actual values depend on the frame, their conservation clearly holds in all frames. We can define a total mass of the system through

$$m_T^2 c^4 = E_T^2 - p_T^2 c^2$$

although this does not in general correspond to the sum of the masses of the particles involved $m_T \neq \sum_i m_i$. Clearly, in the same way as shown above, the total mass is the same in all frames, i.e. is invariant.

The centre-of-mass frame is defined to be the frame where $p_T = 0$, so in this frame $m_T c^2 = E_{\text{cm}} = \sqrt{s}$. Hence, the total mass of the system is often called the centre-of-mass energy and its square is usually denoted by $s$. The invariance of this quantity is very useful, as shown in the examples below.

For a decay of a particle to e.g. three daughter particles, then the outgoing daughter energies and momenta can be added to find $m_T$.

As this is the same in all frames, including the centre-of-mass frame, then by definition, $m_T = m_X$, the mass of the particle decaying. Hence the particle can be identified from its decay product energies and momenta in any frame.

Furthermore, consider a reaction caused by a particle of mass $m_1$ and energy $E_1$ hitting a stationary particle of mass $m_2$.

The total energy $E_T = E_1 + m_2 c^2$ and total momentum magnitude is $p_T c = \sqrt{E_T^2 - m_1^2 c^4}$. The centre-of-mass energy is therefore

$$s = m_T^2 c^4 = E_T^2 - p_T^2 c^2 = E_1^2 + 2E_1 m_2 c^2 + m_2^2 c^4 - E_1^2 + m_1^2 c^4 = 2E_1 m_2 c^2 + m_1^2 c^4 + m_2^2 c^4$$

Hence, we do not need to boost to the centre-of-mass frame to calculate $E_{\text{cm}}$ as it is an invariant. Alternatively, we often want a particular centre-of-mass energy, e.g. to create a new particle, so rearranging gives the required particle energy

$$E_1 = \frac{s - m_1^2 c^4 - m_2^2 c^4}{2m_2 c^2}$$

This frame is often called the fixed target frame as experiments were historically often done by colliding a beam of particles with a stationary target material.
We could consider the reaction in the centre-of-mass, where both particles have equal but opposite momenta. The energies then simply add to give

\[ E'_1 + E'_2 = \sqrt{s} \]

Because it is the centre-of-mass, we have equal momentum magnitudes, so

\[ E'^2_1 - m_1^2c^4 = E'^2_2 - m_2^2c^4 \]

which can be rearranged, using the difference of squares, to give

\[ E'^2_1 - E'^2_2 = (E'_1 + E'_2)(E'_1 - E'_2) = (m_1^2 - m_2^2)c^4 \]

so

\[ E'_1 - E'_2 = \frac{(m_1^2 - m_2^2)c^4}{\sqrt{s}} \]

Summing the two equations gives

\[ 2E'_1 = \sqrt{s} + \frac{(m_1^2 - m_2^2)c^4}{\sqrt{s}} \]

so

\[ E'_1 = \frac{s + (m_1^2 - m_2^2)c^4}{2\sqrt{s}} \]

and similarly for \( E'_2 \). It is instructive to compare these two results, particularly for the case when \( \sqrt{s} \gg m_{1,2}c^2 \). Then in the fixed target frame

\[ E_1 \approx \frac{s}{2m_2c^2} = \frac{\sqrt{s}}{2} \frac{\sqrt{s}}{m_2c^2} \]

while in the centre-of-mass frame

\[ E'_1 \approx \frac{s}{2\sqrt{s}} = \frac{\sqrt{s}}{2} \]

The former is clearly bigger by a factor \( \sqrt{s}/m_2c^2 \gg 1 \). For example, to make a \( Z \) particle, mass 91 GeV/c\(^2\) by colliding positrons and electrons, both with mass 0.511 MeV/c\(^2\), then we need \( E_{\text{cm}} = \sqrt{s} = 91 \) GeV. Hence, in the centre-of-mass frame, the beam energy needed for the positrons and electrons is \( E'_{1,2} = \sqrt{s}/2 = 45.5 \) GeV. However, if the positrons are collided with stationary electrons, then they need an energy in the fixed target frame of \( E_1 \approx s/2m_ec^2 = 8.1 \) TeV. This is much higher and the rest of the energy is “wasted” as kinetic energy of the \( Z \) particle; it must be moving so that momentum overall is conserved. In the centre-of-mass, the \( Z \) is at rest.

Clearly, it is much easier to make the beam energies needed in the centre-of-mass, so most modern experiments have colliding beams and the experiment frame coincides with the centre-of-mass frame.
Spin

If all particles were spin 0, then the above would cover all the important quantities. However, spin is an additional complication which we need to consider. In quantum mechanics, the spin vector $S$ (or in fact any angular momentum) is quantised both in terms of its length and its components. For spin $s$, where $s$ is an integer, the total length of the spin vector is $\sqrt{s(s+1)}\hbar$. Note, it is meaningless to say the length is negative; $s \geq 0$. For the components along any axis, e.g. $z$, quantum mechanics says that the component eigenvalues can be

$$s_z = -s\hbar, -(s-1)\hbar, -(s-2)\hbar, \ldots, (s-2)\hbar, (s-1)\hbar, s\hbar$$

which therefore has $2s + 1$ possible values. The immediate question which arises is which axis is a sensible choice? To discuss this, we need to think about angular momentum in general.

The orbital angular momentum is

$$L = r \times p$$

and the total angular momentum is

$$J = L + S$$

The isotropy of space means that the total angular momentum $J$ is always conserved. This is again a fundamental principle at the same level as energy and momentum conservation. However, we cannot say that either the orbital or spin angular momentum separately will be conserved. Consider a free particle, i.e. one which feels no forces, which therefore has a constant momentum $p$. By definition, the orbital angular momentum $L$ must be perpendicular to $p$ as it is made from a cross product involving $p$. Hence, there is no component of $L$ along $p$, i.e. $L \cdot p = 0$. In contrast, $S$ can be in any direction relative to $p$.

Hence, the component of $J$ along $p$ is equal to $S$ along $p$, i.e. $J \cdot p = S \cdot p$. Since $J$ is conserved, then so are any of its components so $S$ along $p$ is always constant. This quantity is called the helicity

$$h = \frac{S \cdot p}{|p|}$$
Because of this, the axis along which we resolve the spin for a free particle is usually taken to be the momentum axis. For example, for a spin 1/2 particle, then the only possible values of helicity are $h = \pm 1/2$.

Note, helicity (and indeed momentum) are only constant for free particles. In a reaction, particles can (and do) change their momentum and/or helicity. One of the things we study in particle physics is how reactions change these quantities.