Nuclear and Particle Physics - Lecture 20
The shell model

1 Introduction

It is apparent that the semi-empirical mass formula does a good job of describing trends but not the non-smooth behaviour of the binding energy. For this, we need to go to a very different model of the nucleus, which is based on quantum energy levels. It is suprising that such a radically different picture can be describing the same physical system but we shall see that several properties of nuclei are well described by this model.

2 Magic numbers

A closer look at the discrepancies from the semi-empirical mass formula is in order. We saw there were particular values of Z and N for which the nuclei had a higher binding energy than would be expected. These strongly bound states occur when Z or N have one of a set of so-called "magic numbers". In fact, even for $A < 20$, where the semi-empirical mass formula is not valid, it is apparent that certain nuclei, e.g. ${}^4_2$He with a binding energy of 28.3 MeV, are much more strongly bound that their neighbours, e.g. there are no bound $A = 5$ nuclei. The magic numbers which are observed over the whole range of nuclei are

$$2, 8, 20, 28, 50, 82, 126$$

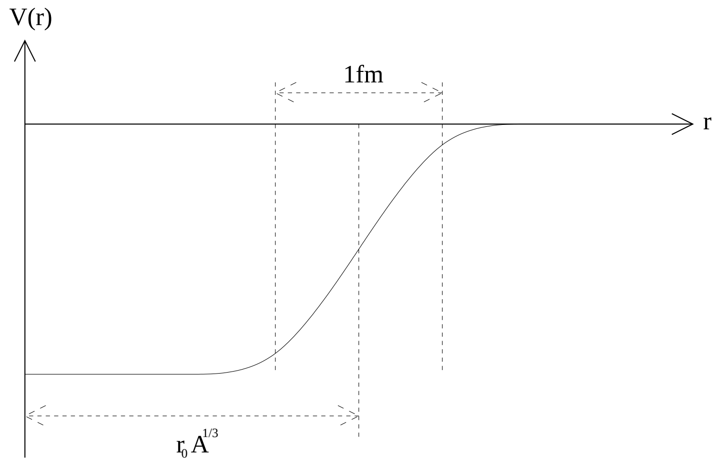
Some nuclei have both Z and N at magic numbers, such as ${}^4_2$He ($Z = 2$, $N = 2$) and the most common isotope of lead, ${}^{208}_{82}$Pb ($Z = 82$, $N = 126$); these are called doubly-magic and are correspondingly even more strongly bound.

The shell model says these magic numbers correspond to filling a quantum energy level, so giving a particularly well-bound nucleus. The magic nuclei are therefore equivalent to the inert gases (helium, neon, argon, etc.) in chemistry. While this provides a qualitative explanation, we still need to understand why the magic numbers have the values they do.

3 Nuclear potentials

Ideally, we would write down the Schrödinger equation for the nuclear force potential and solve it to calculate the energy levels, as done for the hydrogen atom. However, this is not as simple as for hydrogen, for two reasons. Firstly, the potential for the nuclear force is much more complicated than the $1/r$ for hydrogen. Secondly, it is not a central potential in which are nucleons move independently; there is no central object corresponding to the proton in hydrogen but each nucleon feels the force from the others.

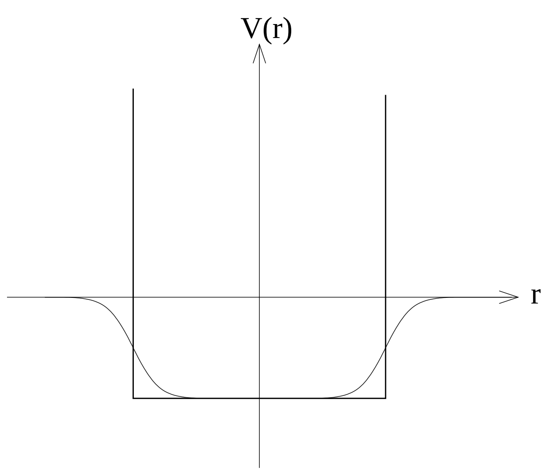
Hence, we need to make a physical guess for a reasonable potential and compare with the observed magic numbers. We will consider each nucleon as moving in a potential resulting from the average of the interactions with all the other nucleons. What would this potential look like? We already saw the short range force means a nucleon is bound to all its nearest neighbours by an equal contribution to the binding energy for each nucleon. Inside the nucleus, the number of nearest neighbours is equal in all directions so the net force on any nucleon is in fact zero. Thus the effective potential is constant within the nucleus and the constant value must be negative to keep the nucleon bound. Outside the nucleus, more than a few fermis away, the short range nuclear force will have died off, so again there will be no force and hence a constant effective potential, which we can take as zero. Finally, as stated previously, the nucleons near the surface have only nearest neighbour forces into the nucleus as they are missing the nearest neighbours outside. Hence, they do have a net inwards force and so a rising potential as the radius increases. This change to the potential takes place over a distance of order the nuclear force, so around 1 fm. Hence, we would guess an effective potential would look like

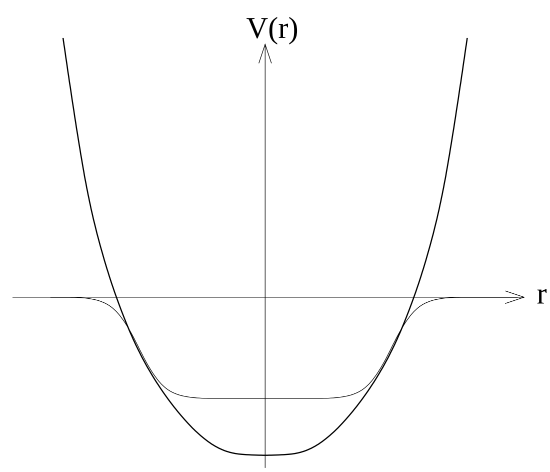


This is called the Saxon-Woods potential and is often mathematically expressed as

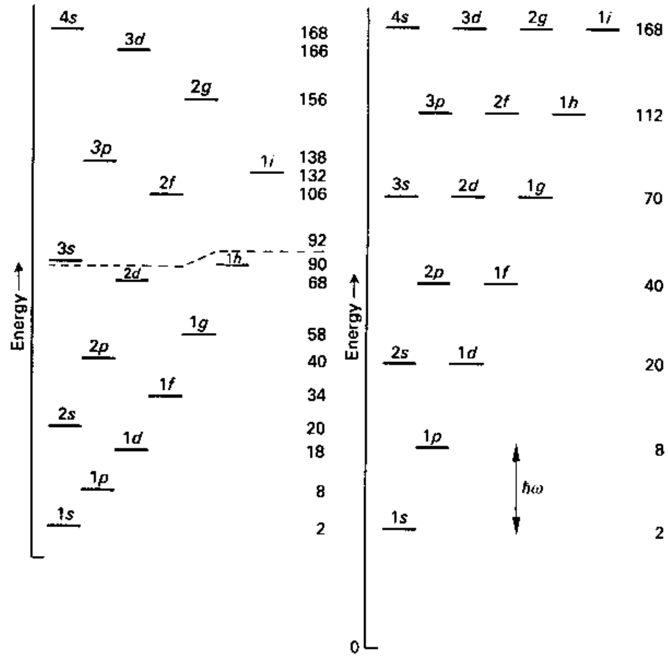
$$V(r) = -\frac{V_0}{1 + e^{(r-a)/d}}$$

where $a \sim r_0 A^{1/3}$ sets the radius and $d \sim 1$ fm the speed with which the potential rises. While it is possible to solve the Schrödinger equation for this potential, it is not trivial. To give a feel for the results, we can look at some simpler cases, such as an infinite square well or a simple harmonic oscillator.





These levels can be calculated more easily and look like the following



What do these predict for the magic numbers? Each state has $2l+1$ values of l_z and due to the nucleon spin, each can take two protons (and also two neutrons) in the two s_z states. Hence, the number of protons (or neutrons) in an l state is $2(2l+1) = 4l+2$.

l	$4l+2$
0	2
1	6
2	10
3	14

Hence, the first magic number of 2 corresponds to filling the first state in both cases. The next magic number is 8, which is the total number of nucleons which fills the first two states, again in either case. The other numbers given by completing the levels are shown in the diagrams above. They both give 20 but then start to disagree with the measured values for the magic numbers. Hence, we can reproduce the first few but not the higher values. You may think this is just a question of tweaking $V(r)$ to arrange the states to be just right, but it turns out it is not possible to get all the correct magic numbers by this method.

4 Spin-orbit coupling

A new term is needed in the potential and this is a spin-orbit coupling, where the energy is $\propto \boldsymbol{l}.\boldsymbol{s}$, just as happens in atomic physics. This has the effect of splitting some of the $4l+2$ degeneracy and giving new energy levels. We previously said each l state has $2l+1$ values of l_z and $2s+1 = 2$ values of s_z. These could equally well be described by total angular momentum j and j_z, rather than l_z and s_z. For a given l, then there are two values of j, namely $l \pm 1/2$ and these have $2j+1 = 2(l \pm 1/2) + 1 = 2l+2$ and $2l$ values of j_z, summing to $4l+2$ in total, as required. Without a spin-orbit coupling, both values of j have the same energy and so are totally degenerate. However, a spin-orbit coupling splits the two j values but leaves the

j_z degeneracy in each one (as is required by the isotropy of space). To see how this works, we can use the same trick as we used for the hyperfine splitting in the mesons. The total angular momentum is

$$\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$$

so squaring gives

$$j^2 = l^2 + s^2 + 2\boldsymbol{l}.\boldsymbol{s}$$

Rearranging, then

$$\boldsymbol{l}.\boldsymbol{s} = \frac{1}{2}\left[j^2 - l^2 - s^2\right]$$

In terms of eigenvalues, this is

$$\langle \boldsymbol{l}.\boldsymbol{s} \rangle = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

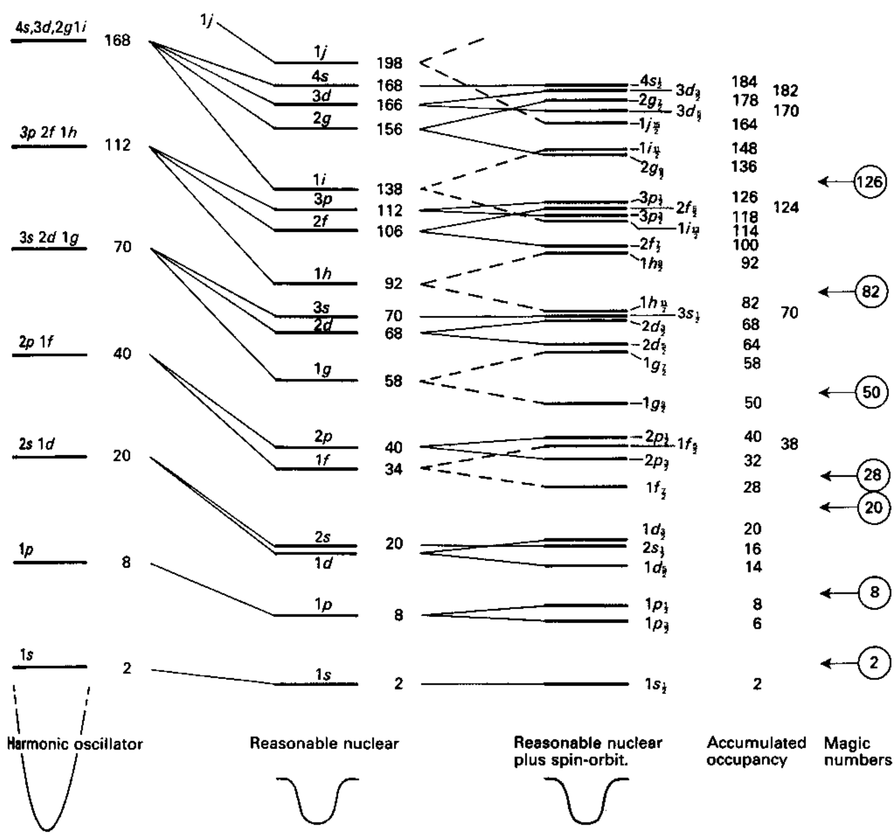
showing that this term does indeed depend on the value of j. Since $j = l \pm 1/2$, then for $l + 1/2$, this gives

$$\langle \boldsymbol{l}.\boldsymbol{s} \rangle = \frac{\hbar^2}{2}\left[(l+1/2)(l+3/2) - l(l+1) - s(s+1)\right] = \frac{\hbar^2}{2}\left[l^2 + 2l + 3/4 - l^2 - l - 3/4\right] = \frac{\hbar^2}{2} l$$

while for $l - 1/2$, it gives

$$\langle \boldsymbol{l}.\boldsymbol{s} \rangle = \frac{\hbar^2}{2}\left[(l-1/2)(l+1/2) - l(l+1) - s(s+1)\right] = \frac{\hbar^2}{2}\left[l^2 - 1/4 - l^2 - l - 3/4\right] = -\frac{\hbar^2}{2}(l+1)$$

The effect of applying this splitting to the Saxon-Woods potential is shown below



This model can also correctly predict the spins and parities of many nuclei where there is a single unpaired nucleon either alone in a state or missing from a completed state. It works particularly well for Z or N close to magic. All even-even nuclei are $J^P = 0^+$. If Z or N are both even and one corresponds to a completed level, then adding one extra of this type of nucleon means the total spin of the nucleus must be the angular momentum of this final nucleon, as must its parity. E.g. ${}^{17}_{8}$O has $Z = 8$ and $N = 9$, so the final neutron must be in the next state above the level which gives the magic number 8. This is a $1d_{5/2}$ level and so has $j = 5/2$ and $l = 2$, which gives $P = (-1)^l = +1$. Hence, this nucleus would be expected to be $J^P = 5/2^+$, as observed. Similarly, removing one nucleon from a filled magic state gives a nucleus with total spin exactly opposite to the removed nucleon (as they sum to give zero), i.e. the same j value but opposite j_z, and also the same parity (as they multiply to give +1). This means it has the same quantum numbers as the unfilled state. Hence, for example ${}^{15}_{8}$O, with $Z = 8$ and $N = 7$, would have the properties of the $1p_{1/2}$ state, which has $l = 1$ and hence $P = (-1)^l = -1$, and so would be expected to be $J^P = 1/2^-$, again as observed.