Laplace transforms

- Once again a practical exposition, not fully mathematically rigorous
- Definition

$F(s) = \int_0^{\infty} f(t).e^{-st}.dt$ NB lower limit of integral = 0 **unilateral LT**

more rigorously $F(s) = \int_{0+}^{\infty} f(t).e^{-st}.dt = {}_{\text{limit } h\to 0}\int_{|h|}^{\infty} f(t).e^{-st}.dt$

[Another variant exists $F(s) = \int_{-\infty}^{\infty} f(t).e^{-st}.dt$ **bilateral LT]**

Unilateral LT convenient for systems where nothing happens before t=0

the inverse Laplace transform is much more complicated mathematically than the Fourier transform,

$f(t) = (1/2\pi j)\int_{c-j\infty}^{c+j\infty} F(s).e^{st}.ds$ $j = \sqrt{-1}$

Cauchy principal value of integral in complex plane

However, this is not generally required in most practical cases. There are many problems where inverse transforms can be found by inspection.

Conventions

•- as for Fourier

f: function to be transformed

F: Laplace transform of f $F = LT[f]$ and inverse $f = LT^{-1}[F]$

Unless specifically stated all functions f(t) are assumed to take the value

$f(t) = 0 \quad t < 0$

not a real constraint for practical problems

Formally, this can always be achieved for any function by multiplying by unit step function u(t)

•Why use the Laplace transform instead of Fourier?

particularly suited for transient problems

some functions don't converge

Fourier response is an integral

sometimes Laplace vs Fourier is just preference

The meaning of s

- In Fourier transforms the complementary variable usually has a clear physical meaning,

  eg if working in time t <=> ω or f

  diffraction in optics, where FTs are used, has a similar relationship between spatial distributions and spatial freqency

- Although Laplace transforms look very similar (and many results can be easily obtained by following methods for deriving FTs), the complementary variable s does not have the same physical significance.

  It is a mathematical method of solving problems using transforms

- Since we spent a significant time on the FT, I will not spend so much time on the details of deriving LTs

  integrals are usually straightforward

  I will discuss only transforms we will need here

Some theorems (compare to FT)

PROVE THEM!!

- **Linearity** $LT[a.f(t)+b.g(t)] = a.F(s) + b.G(s)$

- **Shifting in time**

  $LT[f(t-\Delta t)] = \int_0^\infty f(t-\Delta t).\,e^{-st}.dt = e^{-s\Delta t}F(s)$

- **Translation in s**

  $LT[f(t)e^{-at}] = \int_0^\infty f(t)\,e^{-at}e^{-st}dt = F(s+a)$

- **Convolution**

  $LT[x(t)*y(t)] = X(s)Y(s)$

- **Differentiation**

  $f'(t) = d/dt\{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} F(s).e^{st}.ds\} = \{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} sF(s).e^{st}.ds\}$

  $LT[f'(t)] = sF(s)$

- **Integration**

  $\int_0^t f(t)dt = \int_0^t \{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} F(s).e^{st}.ds\}dt$

  $= \{(1/2\pi j)\int_{c-j\infty}^{c+j\infty} (1/s)F(s).e^{st}.ds\}$

  $LT[\int_0^t f(t)dt] = F(s)/s$

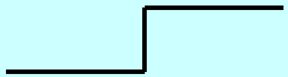
these are results to be remembered (or derived)

Some examples

PROVE THEM!!

(1) $f(t) = e^{-at} \quad t \geq 0$ $\qquad F(s) = \int_0^\infty e^{-at}.e^{-st}.dt = \int_0^\infty e^{-(s+a)t}.dt = 1/(s+a)$

(2) $f(t) = u(t) = 1 \quad t \geq 0$ $\qquad F(s) = \frac{1}{s}$



(3) $f(t) = \delta(t-t_0)$ $\qquad F(s) = \int_0^\infty \delta(t-t_0).e^{-st}.dt = e^{-st_0}$ $\qquad LT[\delta(t)] = 1$

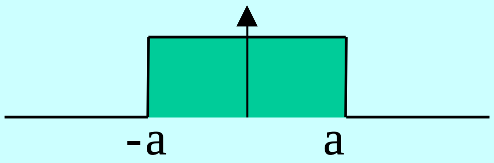
(4) $f(t) = \delta'(t-t_0)$ $\qquad F(s) = se^{-st_0}$ $\qquad LT[\delta'(t)] = s$

(5) $f(t) = 1-e^{-at}$ $\qquad F(s) = \dfrac{a}{s(s+a)}$

(6) $f(t) = ate^{-at}$ $\qquad F(s) = \dfrac{a}{(s+a)^2}$

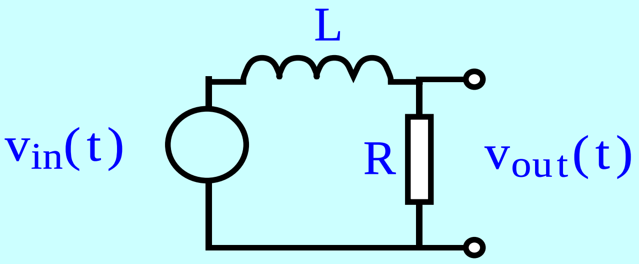
(7) $f(t) = t^n e^{-at}$ $\qquad F(s) = \dfrac{n!}{(s+a)^{n+1}}$

(8) $\Pi(t)$ $\qquad F(s) = 2\sinh(sa)/s$



Problem solving with LT

- Inductor - resistor circuit



$$v_{out}(t) = i(t)R \qquad L\frac{di}{dt}(t) + Ri(t) = v_{in}(t)$$

$$\frac{L}{R}\frac{dv_{out}}{dt}(t) + v_{out}(t) = v_{in}(t)$$

- Take Laplace transform

$$\frac{L}{R}sV_{out}(s) + V_{out}(s) = V_{in}(s)$$

- solution

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{\frac{sL}{R}+1} = \frac{a}{s+a} \qquad a = R/L$$

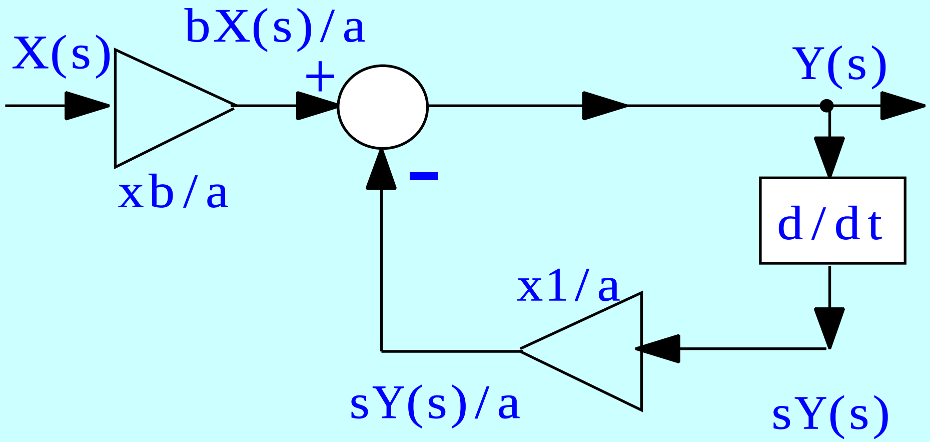
- Example

$$v_{in}(t) = u(t) = \text{unit step} \qquad V_{in}(s) = \frac{1}{s} \qquad V_{out}(s) = \frac{a}{s(s+a)}$$

LT of $1 - e^{-at} = v_{out}(t)$

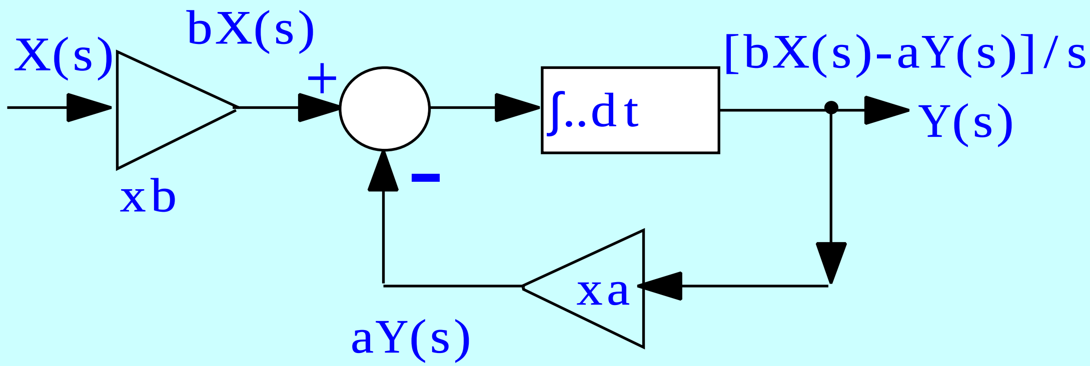
Solution of differential equations

- Solve $\frac{dy(t)}{dt} + ay(t) = bx(t)$ where $x(t)$ = input $y(t)$ = output
- rewrite as $y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$ and $Y(s) = -\frac{1}{a}sY(s) + \frac{b}{a}X(s)$
- system block diagram



- alternatively

$$y(t) = \int_0^t [bx(u) - ay(u)]du$$



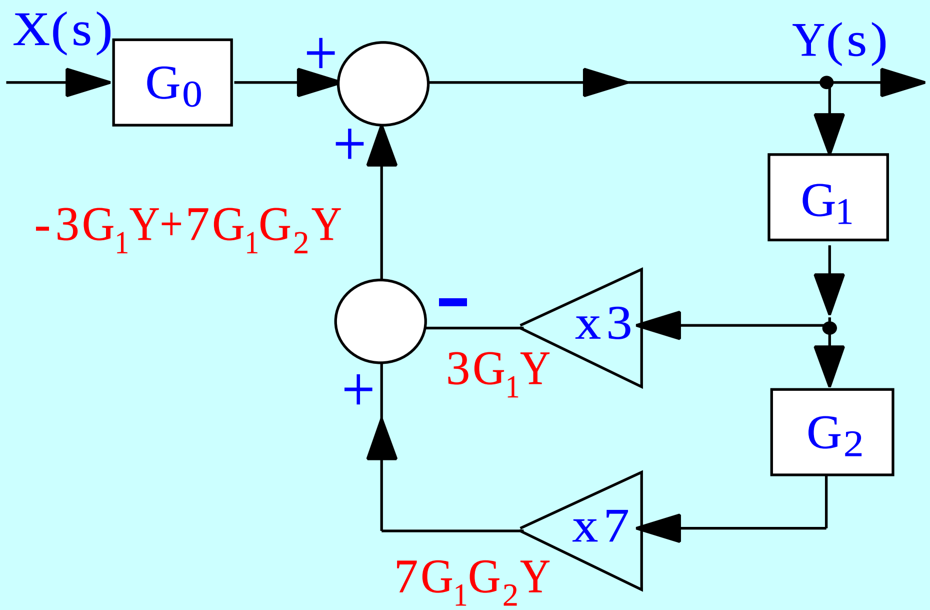
- if $x(t)$ is known, full solution to system response can be found

Example (from 2001 exam)

- (i) derive system transfer function

$$Y = G_0X - 3G_1Y + 7G_1G_2Y$$

$$Y(s) = \frac{G_0X(s)}{1 + 3G_1 - 7G_1G_2}$$



- (ii) G_0 has time domain response $24te^{-2t}$

  G_1 is unity gain differentiator

  G_2 is unity gain integrator

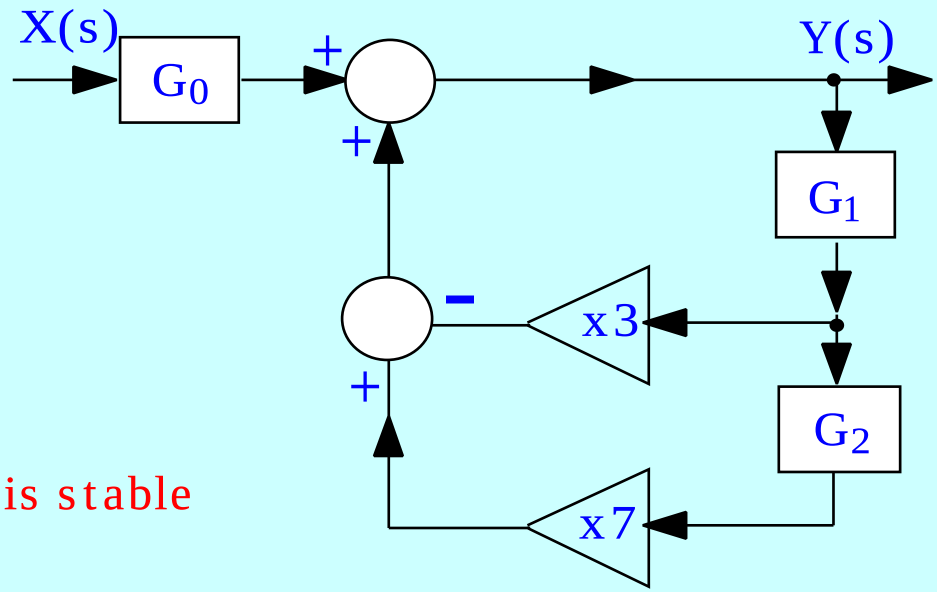
$$G_0(s) = \frac{24}{(s+2)^2} \qquad G_1(s) = s \qquad G_2(s) = \frac{1}{s}$$

$$Y(s) = \frac{24X(s)}{(s+2)^2(1+3s-7)} = \frac{8X(s)}{(s+2)^2(s-2)}$$

- (iii) Is system stable to small perturbations?
- (iv) Find time domain response to step u(t), for t > 0

Stability

$$Y(s) = \frac{8X(s)}{(s+2)^2(s-2)}$$



•System has 2 poles: points where Y(s) -> ∞

at s = +2 and s = -2

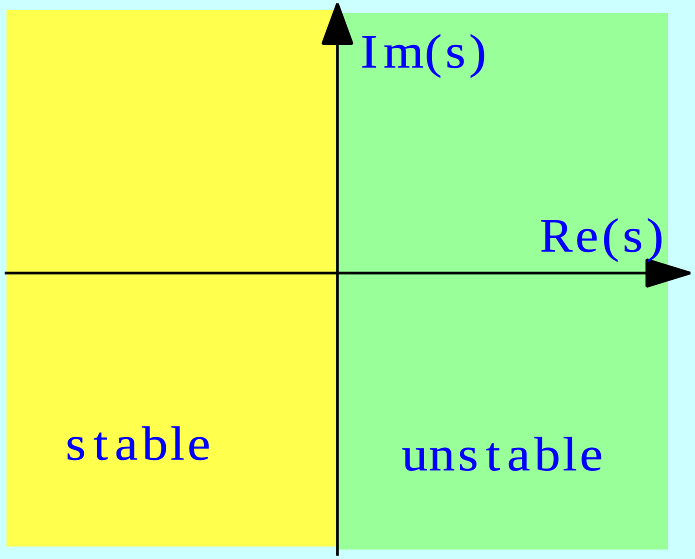
•If all poles are in region where s < 0, system is stable

in Fourier language s = jω

can only have positive frequencies, ie s > 0

so this system is unstable

will see why from solution



•Pole location s could have imaginary part

=> oscillatory solution

Response to step

- $x(t) = u(t) = 1$, for $t > 0$ so $X(s) = 1/s$

$$Y(s) = \frac{8X(s)}{(s+2)^2(s-2)} = \frac{8}{s(s+2)^2(s-2)}$$

$$= \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2} + \frac{D}{(s-2)}$$

- Solve by expressing as partial fractions
- Find A, C, D by taking limit s -> a of $(s+a)^N Y(s)$ N is highest power term
- Find A by multiplying by s

RHS $$\underbrace{\text{limit}}_{s->0} \ldots sY(s) = A + \frac{Bs}{(s+2)} + \frac{Cs}{(s+2)^2} + \frac{Ds}{(s-2)} = A$$

LHS $$\underbrace{\text{limit}}_{s->0} \ldots sY(s) = \frac{8}{(s+2)^2(s-2)} = \frac{8}{4(-2)} = -1$$

$$A = -1$$

- Find C by multiplying by $(s+2)^2$

RHS $$\underbrace{\text{limit}}_{s->-2} \ldots (s+2)^2 Y(s) = A(s+2)^2 + B(s+2) + C + \frac{D(s+2)^2}{(s-2)} = C$$

$$C = 1$$

LHS $$\underbrace{\text{limit}}_{s->-2} \ldots (s+2)^2 Y(s) = \frac{8}{s(s-2)} = \frac{8}{(-2)(-4)} = 1$$

similarly $D = 1/4$

Step response... continued

$$Y(s) = \frac{8X(s)}{(s+2)^2(s-2)} = \frac{8}{s(s+2)^2(s-2)} \qquad = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2} + \frac{D}{(s-2)}$$

•Find B by multiplying by $(s+2)^2$, differentiate, then take limit

RHS

$$\frac{d}{ds}(s+2)^2 Y(s) = \frac{d}{ds}\left[\frac{8}{s(s-2)}\right] = 8\left[\frac{-1}{s^2(s-2)} + \frac{-1}{s(s-2)^2}\right]$$

$$\underbrace{\text{limit}}_{s->-2}\left(8\left[\frac{-1}{s^2(s-2)} + \frac{-1}{s(s-2)^2}\right]\right) = 8\left[\frac{-1}{4(-4)} + \frac{-1}{(-2)(-4)^2}\right] = \frac{3}{4}$$

LHS

$$\underbrace{\text{limit}}_{s->-2} \ldots \frac{d}{ds}(s+2)^2 Y(s) = \frac{d}{ds} B(s+2) = B$$

$$B = \frac{3}{4}$$

•now have the solution in s

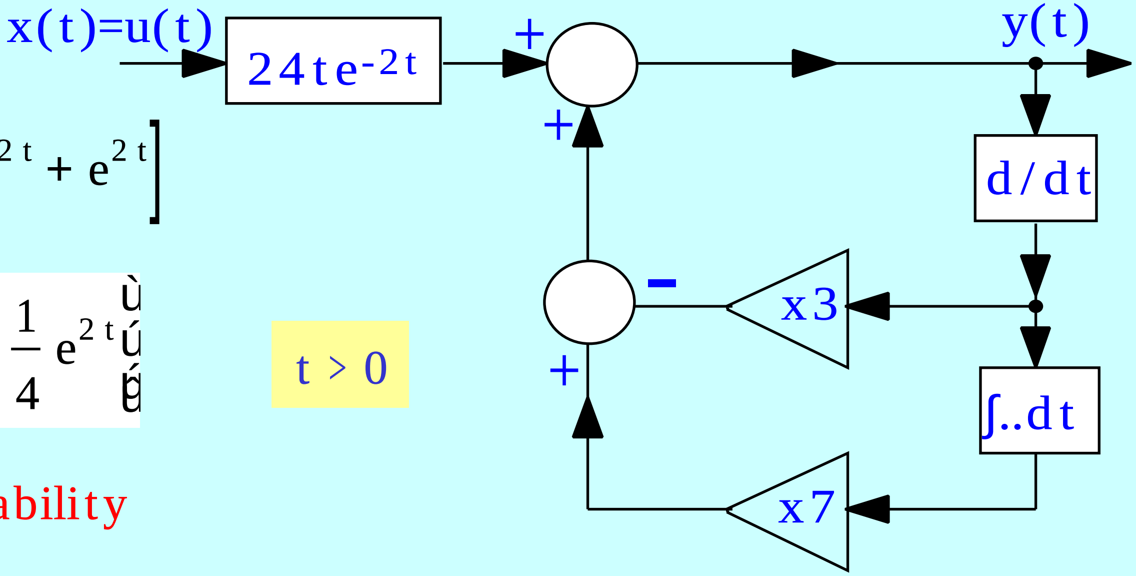
$$Y(s) = \frac{1}{4}\left[\frac{-4}{s} + \frac{3}{(s+2)} + \frac{4}{(s+2)^2} + \frac{1}{(s-2)}\right]$$

Finally… solution

$$Y(s) = \frac{1}{4}\left[\frac{-4}{s} + \frac{3}{(s+2)} + \frac{4}{(s+2)^2} + \frac{1}{(s-2)}\right]$$

•Recall $F(s) = \frac{n!}{(s+a)^{n+1}}$ is LT of $f(t) = t^n e^{-at}$

•and $F(s) = \frac{1}{s}$ is LT of $u(t)$ = unit step



$$y(t) = \frac{1}{4}\left[-4u(t) + 3e^{-2t} + 4te^{-2t} + e^{2t}\right]$$

$$y(t) = -u(t) + \left[\frac{3}{4}e^{-2t} + te^{-2t} + \frac{1}{4}e^{2t}\right]$$

$t > 0$

•Can now see the reason for instability

term with e^{2t}

•By the way: this problem could equally well be solved with Fourier

z transforms

- Laplace transform applies to continuous signals in time domain

  Extend idea to discrete, sampled signals

- from Laplace Transform definition

  $F(s) = \int_0^\infty f(t).e^{-st}.dt,$

  sample waveform f(t) at intervals Δt

  sampled signal

  $f(t) = f(0), f(\Delta t), f(2\Delta t), f(3\Delta t), f(4\Delta t),..., f(n\Delta t), ...$

  We will assume functions for which $f = 0$ for $t < 0$

- transform f(t)

  $F(s) = \sum_{n=0}^{\infty} f(n\Delta t).e^{-sn\Delta t}$

  Define $z = e^{s\Delta t}$

  $F(z) = \sum_{n=0}^{\infty} f(n\Delta t).z^{-n} = \sum_{n=0}^{\infty} f_n.z^{-n}$

  ZT[f] = F(z)

  each term in z^{-1} represents a delay of Δt, ie z^{-n} => delay of $n\Delta t$

Examples

- (1) $f_n = \delta_0$ = 1 0 0 0 0 ...

  $F(z) = 1$

- (2) $f_n = 1$ represents a step function, since f(t) = 0 for all t < 0

  $F(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots + z^{-n} + \ldots$

Should recognise geometric series, or binomial expansion of $(1-x)^{-1}$

$$F(z) = \frac{1}{(1 - z^{-1})}$$

- (3) $f_n = e^{-na}$ $a = \Delta t / \tau$ τ = time constant Δt = sampling interval

  $F(z) = 1 + e^{-a}z^{-1} + e^{-2a}z^{-2} + e^{-3a}z^{-3} + e^{-4a}z^{-4\ldots} \ldots + e^{-na}z^{-n} + \ldots$

$$F(z) = \frac{1}{(1 - e^{-a}z^{-1})}$$

- (4) $f_n = 1 - e^{-na}$

$$F(z) = \frac{1}{(1 - z^{-1})} - \frac{1}{(1 - e^{-a}z^{-1})} = \frac{z^{-1}(1 - e^{-a})}{(1 - z^{-1})(1 - e^{-a}z^{-1})}$$

Digital filters

- What is the output if every previous input sample is summed with weight e^{-na}?

  ie compute $g_m = \sum_n^m e^{-na} f_n$

- Convolution in time, so becomes z-transform multiplication $G(z) = H(z)F(z)$

$$H(z) = ZT[e^{-na}] = \frac{1}{(1 - e^{-a}z^{-1})} \qquad G(z) = \frac{F(z)}{(1 - e^{-a}z^{-1})}$$

$$F(z) = (1 - e^{-a}z^{-1})G(z) = G(z) - G(z)e^{-a}z^{-1}$$

$$f_n = g_n - e^{-a}g_{n-1} \qquad \text{or} \qquad g_n = f_n + e^{-a}g_{n-1}$$

- ie - Latest value of output sampled waveform

  = current input sample + previous output sample x e^{-a}

- Impulse response corresponding to H(z)?

  $h(t) = e^{-n\Delta t/\tau}$ which is impulse response of Low Pass Filter (Problems 2, No 8)

- Conclusion

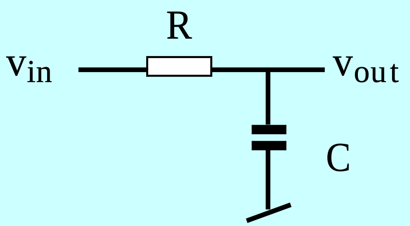
  Low pass digital filter can be made using just two samples $g_n = f_n + e^{-a}g_{n-1}$

  well suited for simple digital processor operation

Step response of previous digital filter

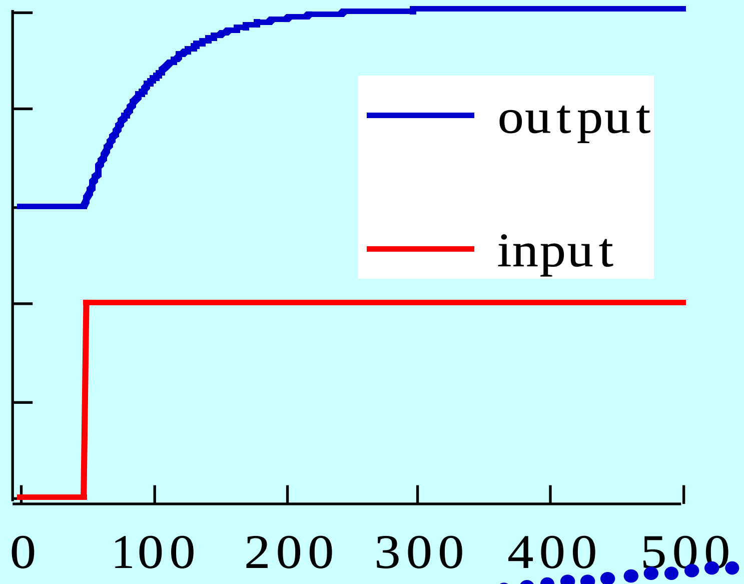
- To be more exact

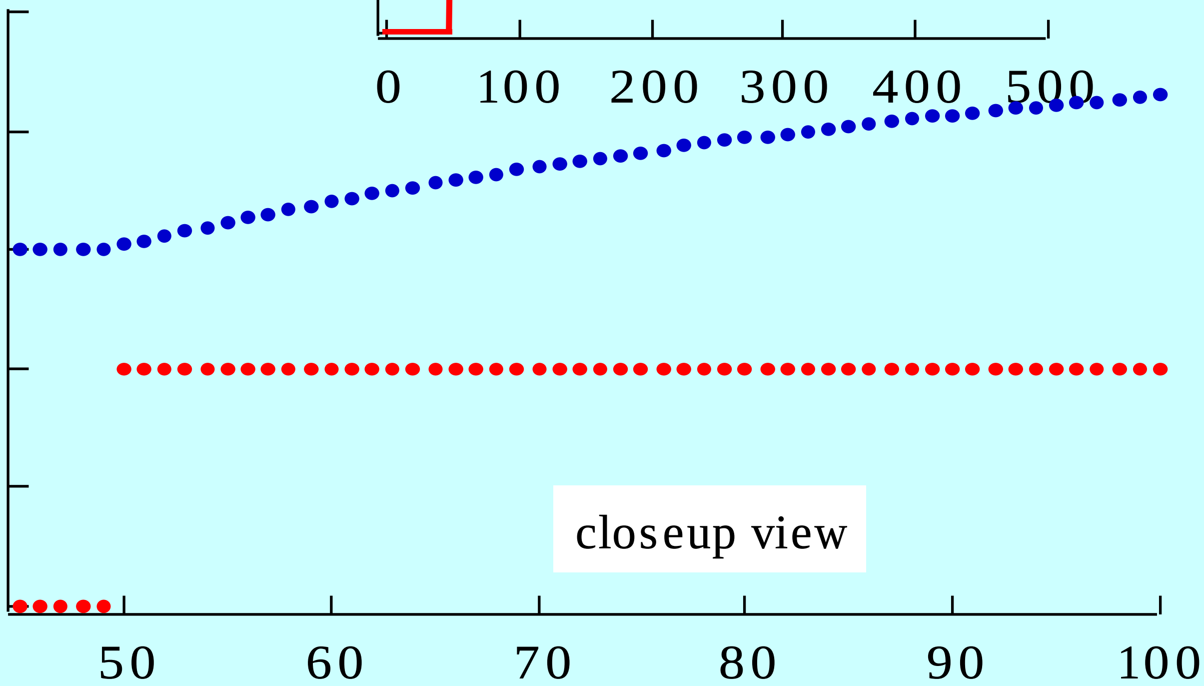
Impulse response of Low Pass filter



$$h(t) = \frac{1}{\tau} e^{-t/\tau}$$

$$g_n = \frac{f_n}{\tau} + e^{-a} g_{n-1}$$





Deconvolution

- Suppose a signal has been filtered by a system with a known response

How to recover the input signal from the samples?

In t: input = f output = g, filter impulse response = h

In z: F(z) G(z) and H(z)

Since g(t) = f(t)*h(t), then $G(z) = F(z)H(z)$

so to recover input $F(z) = H^{-1}(z)G(z)$

- Low pass filter again

$$H(z) = \frac{1}{(1 - e^{-a}z^{-1})}$$

Inverse filter $H^{-1}(z) = (1 - e^{-a}z^{-1})$

$$f_n = g_n - e^{-a}g_{n-1}$$

terms in z^{-1} identify which delayed samples to use

- This time g_n are the measured samples, f_n the result of digital processing

An example of a deconvolution filter

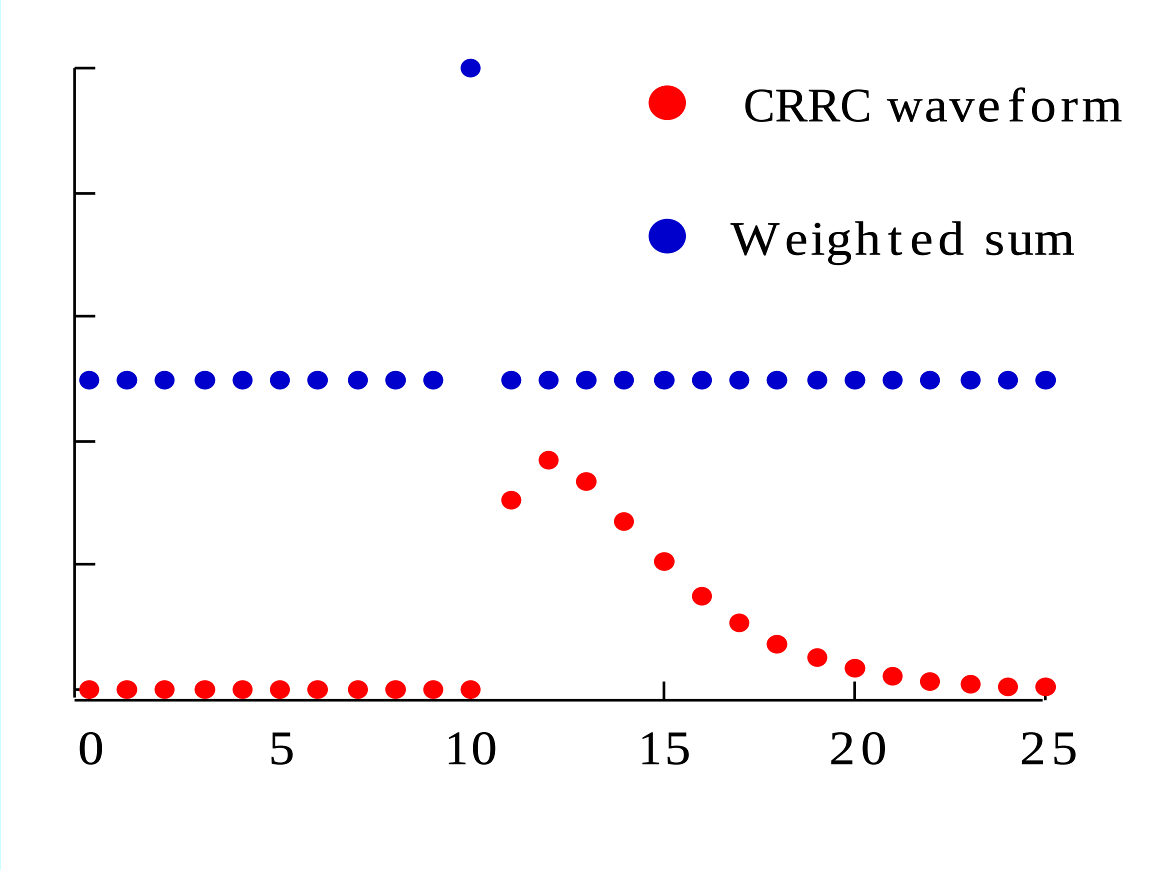
- Integrator + CR-RC bandpass filter waveform

  form weighted sum of pulse samples

  $g_n = w_1.f_{n+1} + w_2.f_n + w_3.f_{n-1}$

  for correct choice of w_i

  (Problems 6)



- Note g_n needs f_{n+1}

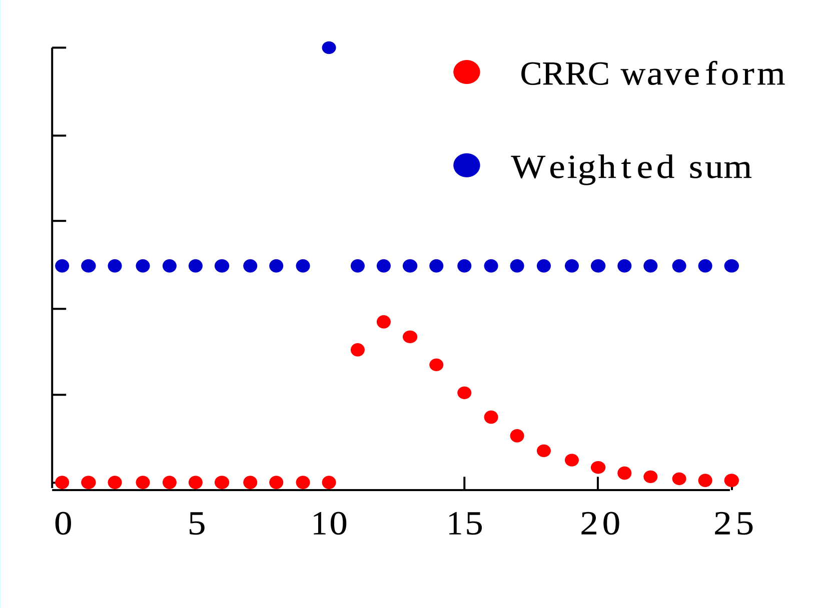
  doesn't violate causality if data are digital, in storage - or could simply delay output

  in applications such as image processing, causality does not apply

CMS experiment at Large Hadron Collider

- uses this deconvolution filter

  implemented in CMOS IC



beam crossings at 40MHz (Δt = 25ns)

many events per crossing

small number of weights

implemented as analogue calculation

process only data which are to be read out