Fourier transforms

- This is intended to be a practical exposition, not fully mathematically rigorous

  ref *The Fourier Transform and its Applications* R. Bracewell (McGraw Hill)

- Definition

  $F(\omega) = \int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt \qquad \omega = 2\pi f$

  $f(t) = \int_{-\infty}^{\infty} F(\omega).e^{j\omega t}.df = (1/2\pi)\int_{-\infty}^{\infty} F(\omega).e^{j\omega t}.d\omega$

should know these !

other definitions exist

- Conventions

  f: function to be transformed

  F: Fourier transform of f F = FT[f]

  so inverse transform is $f = FT^{-1}[F]$

there will be a few exceptions
to upper/lower case rule

What is the importance?

- widely used in many branches of science

  some problems solved more easily by a transform to another domain

  eg algebra just becomes simpler but sometimes understanding too..

  in instruments decomposition of signals in the time domain into frequency, and vice versa, is a valuable tool

- this will be the main interest here (ie t & f)

- Both time development $f(t)$ and spectral density $F(\omega)$ are observables

- Should note that not all functions have FT

  Formally, require

  (i) $\int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt < \infty$

  (ii) f(t) has finite maxima and minima within any finite interval

  (iii) f(t) has finite number of discontinuities within any finite interval

Impulse

- A common signal in physics is an impulse - a la Dirac

ie $\delta(t-t_0) = 0 \quad t \neq t_0$

$\int_{-\infty}^{\infty} \delta(t-t_0) = 1$ or if range of integration includes t_0

- Such a definition is comparable to many detector signals

eg. a scintillation detector measures ionisation of a cosmic ray particle

a pulse from a photomultiplier converts light into electrical signal

the signal is fast (very short duration, typically ~ns)

the total charge in the pulse is fixed

other examples: fast laser pulse, most ionisation

even if the signal is not a "genuine" impulse, it can be considered as a sum of many consecutive impulses

or the subsequent processing may be long in comparison with the signal duration for the approximation to be valid

FT of impulse

- $F(\omega) = \int_{-\infty}^{\infty} \delta(t).e^{-j\omega t}.dt = 1$

  ie an impulse contains a uniform mixture of **all** frequencies

  an important general comment is that short duration pulses have a wide range of frequencies, as do pulses with fast edges (like steps). Real instruments do not support infinite frequency ranges.

- Note on inverting FTs

  $f(t) = \int_{-\infty}^{\infty} F(\omega).e^{j\omega t}.df$

  $= (1/2\pi)\int_{-\infty}^{\infty} F(\omega).\, e^{j\omega t}.d\omega$

  Many inversions are straightforward integrations

  others need care

  eg inverse of δ function $(1/2\pi)\int_{-\infty}^{\infty} 1.\, e^{j\omega t}.d\omega$

  $= (1/2\pi)[e^{j\omega t}/jt\ ]_{-\infty}^{\infty}$???

  often simpler to recognise the function from experience (practice!)

Some theorems

$F(\omega) = FT[f(t)] = \int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt$

•Linearity $FT[a.f(t)+b.g(t)] = a.F(\omega) + b.G(\omega)$

•Translation in time (Shift theorem)

$FT[f(t+t_0)] = \int_{-\infty}^{\infty} f(t+t_0).e^{-j\omega t}.dt$

$= \int_{-\infty}^{\infty} f(u).e^{-j\omega(u-t_0)}.du$

$= e^{j\omega t_0}\int_{-\infty}^{\infty} f(u).e^{-j\omega u}.du$

$= e^{j\omega t_0} F(\omega)$

different frequency components of waveform suffer different phase shifts to maintain pulse shape

•Similarity - scale by factor a > 0

$FT[f(at)] = \int_{-\infty}^{\infty} f(at).e^{-j\omega t}.dt = \int_{-\infty}^{\infty} f(u).e^{-j\omega u/a}.du/a = \int_{-\infty}^{\infty} f(u).e^{-j(\omega/a)u}.du/a$

$= (1/|a|)F(\omega/a)$

compression of time scale= expansion of frequency scale

•Modulation

$FT[f(t)\cos\alpha t] = (1/2)\int_{-\infty}^{\infty} f(t).[e^{j\alpha t} + e^{-j\alpha t}]. e^{-j\omega t}dt$

$= (1/2)\{\int_{-\infty}^{\infty} f(t).e^{-j(\omega-\alpha)t}.dt + \int_{-\infty}^{\infty} f(t).e^{-j(\omega+\alpha)t}.dt\}$

$= (1/2)\{F(\omega-\alpha) + F(\omega+\alpha)\}$

and tricks

- sometimes the symmetry can be exploited to ease calculation

$F(\omega) = \int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt$ $\quad 2\pi f(t) = \int_{-\infty}^{\infty} F(\omega).e^{j\omega t}.d\omega$ $\quad$ FT pair

interchange ω *and* t => $2\pi f(\omega) = \int_{-\infty}^{\infty} F(t).e^{j\omega t}.dt$

so $\int_{-\infty}^{\infty} F(t).e^{-j\omega t}.dt = 2\pi f(-\omega)$

example

$FT[\delta(t)] = 1$ so $FT[1] = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$

- We will very often be dealing with real functions in time

ie. $f(t) = Re[f(t)] + j\ Im[f(t)] = Re[f(t)]$

so complex conjugate $f^*(t) = f(t)$

then $F(-\omega) = F^*(\omega)$

Some examples (i)

(1) $f(t) = 0 \quad t < 0$

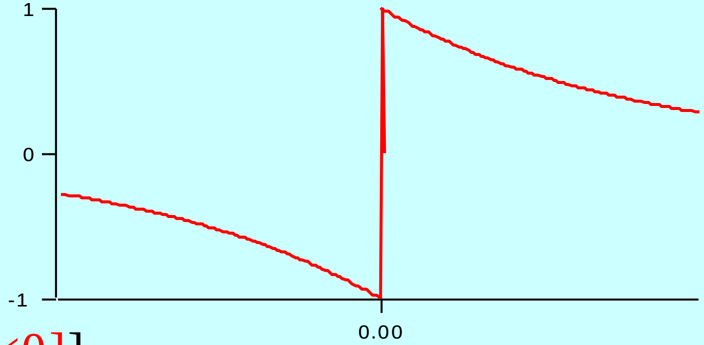
$= e^{-at} \quad t \geq 0$

$F(\omega) = \int_0^\infty e^{-at}.e^{-j\omega t}.dt = \int_0^\infty e^{-(j\omega+a)t}.dt = 1/(j\omega+a)$

(2) $f(t) = e^{-a|t|}$

$F(\omega) = \int_{-\infty}^0 e^{at}.e^{-j\omega t}.dt + \int_0^\infty e^{-at}.e^{-j\omega t}.dt$

$= -1/(j\omega-a) + 1/(j\omega+a) = 2a/(\omega^2+a^2)$



(3) $f(t) = 0 \quad t < 0$

$= 1 \quad t \geq 0$

rewrite as $\lim_{a \to 0} (1/2)[1 + e^{-at}$ [$t \geq 0$] $- e^{at}$ [$t<0$]]

$F(\omega) = \lim_{a \to 0} (1/2)[2\pi\delta(\omega) + 1/(j\omega+a) + 1/(j\omega-a)]$

$= \pi\delta(\omega) + 1/j\omega$

$= 1/j\omega \qquad \omega > 0$

this function is often called H(t)

Some examples (ii)

(4) $f(t) = 0 \quad t < 0$

$= 1 - e^{-at} \quad t \geq 0$

$F(\omega) = a/[j\omega(j\omega+a)] \qquad \omega > 0$

(5) $f(t) = 0 \quad t < 0$

$= ate^{-at} \quad t \geq 0$

$F(\omega) = a/(j\omega+a)^2$

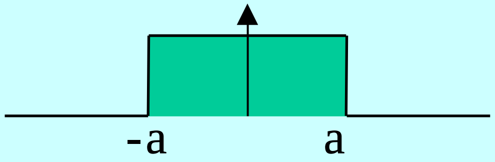
(6) $f(t) = \exp(-a^2t^2)$

$F(\omega) = (\sqrt{\pi}/a)\exp(-\omega^2/4a^2)$

(7) top-hat function $\Pi(t)$

$f(t) = 1 \quad -a < t \leq a$

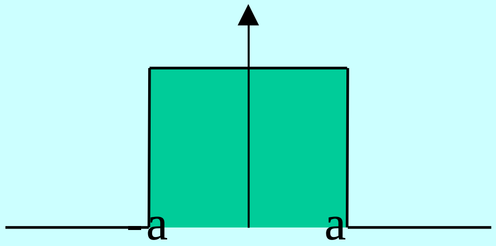
$= 0 \quad$ elsewhere

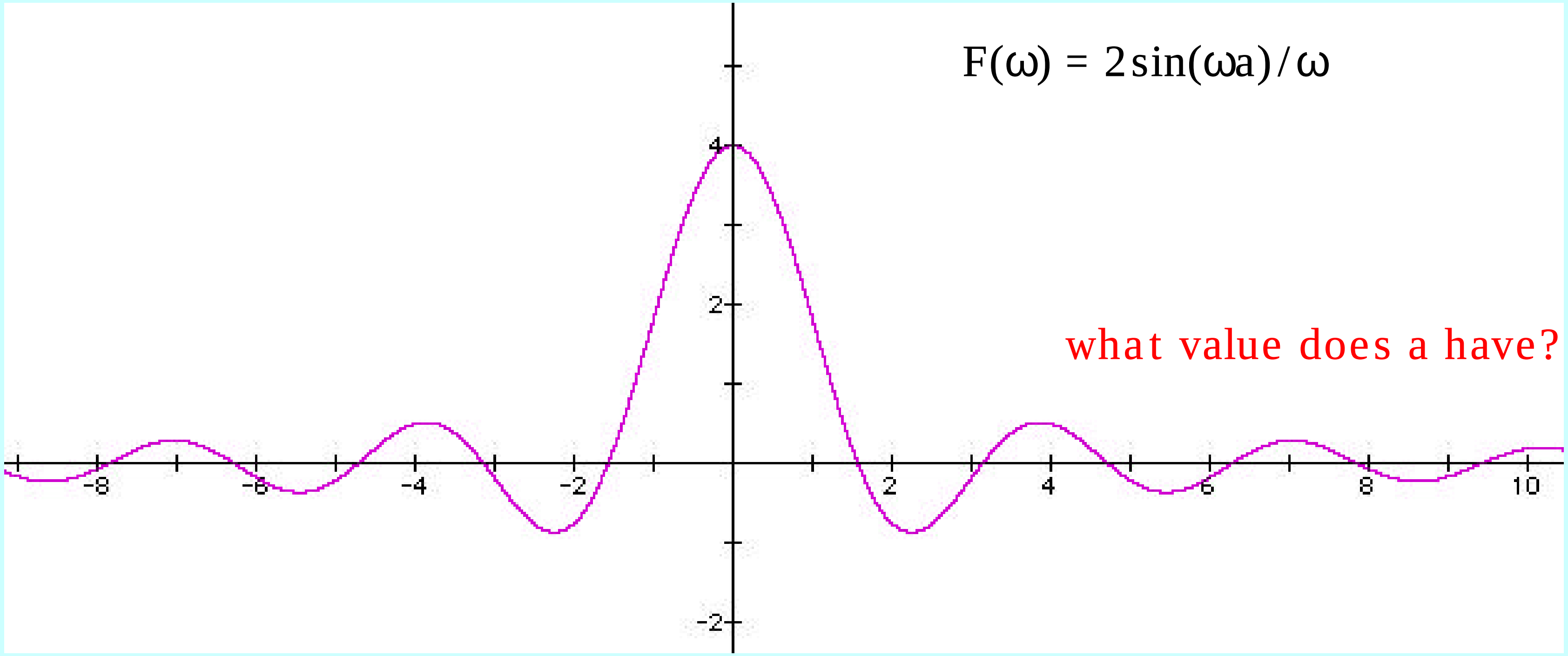


$F(\omega) = 2\sin(\omega a)/\omega$

Fourier pairs

- top hat function





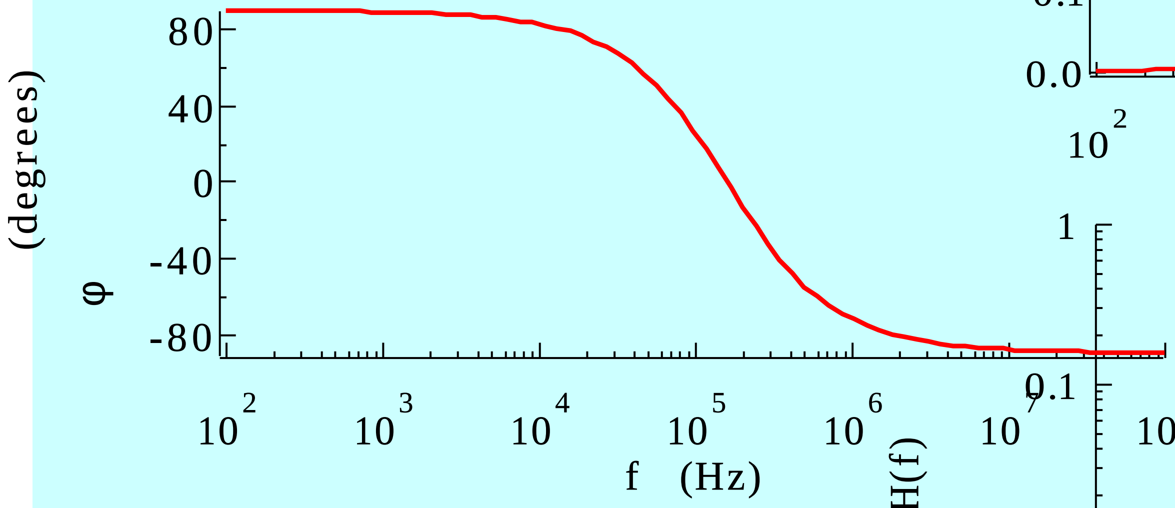
Bandpass filter

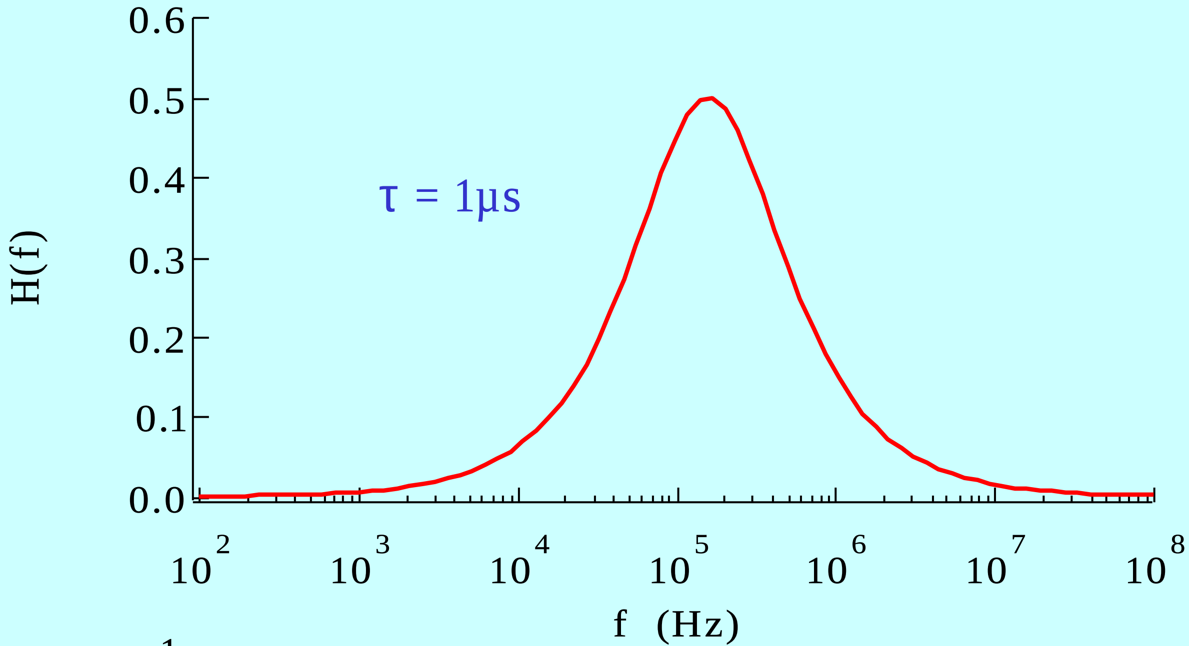
- Low pass + high pass filters

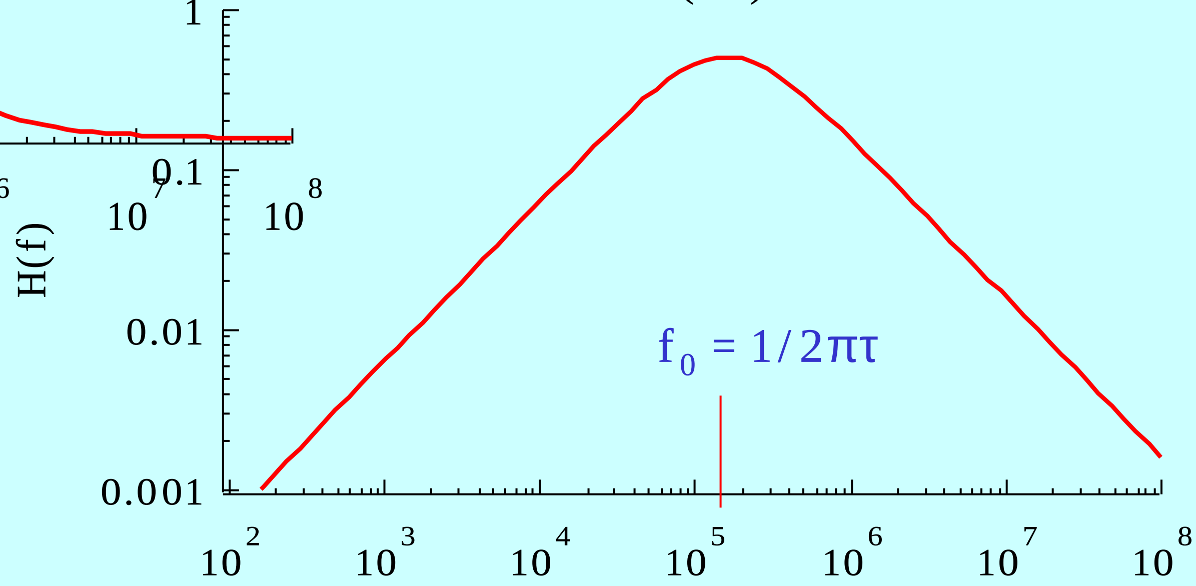
  equal time constants

  are often chosen

$$F(\omega) = j\omega\tau/(1+j\omega\tau)^2$$





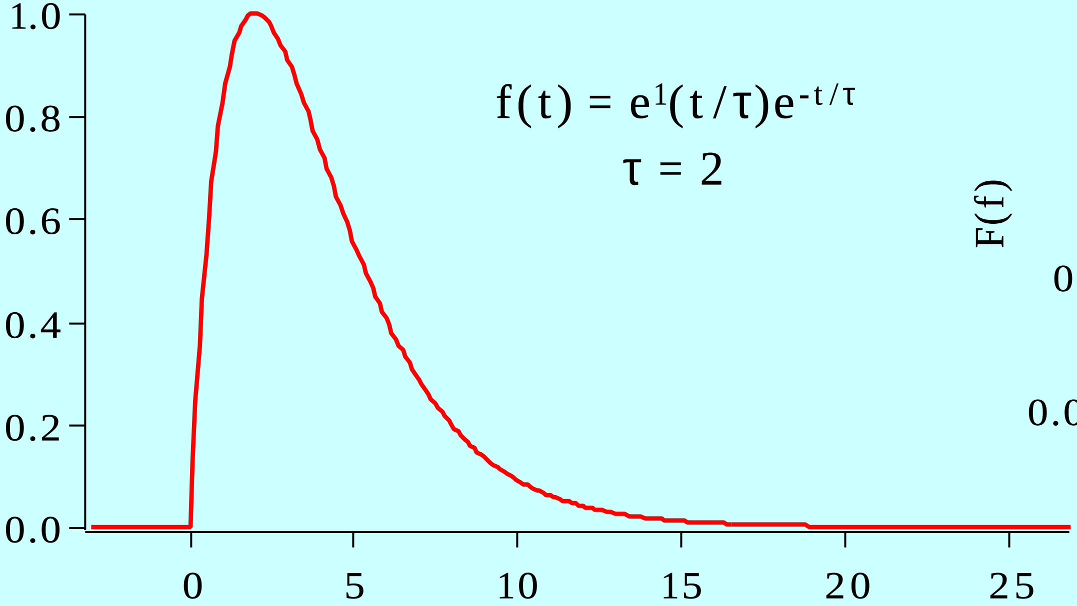


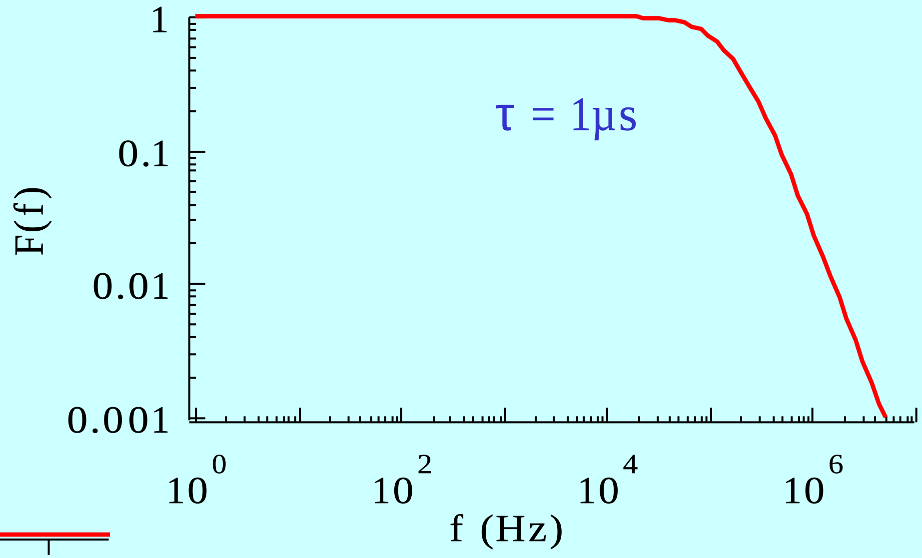
Integrator + Bandpass filter

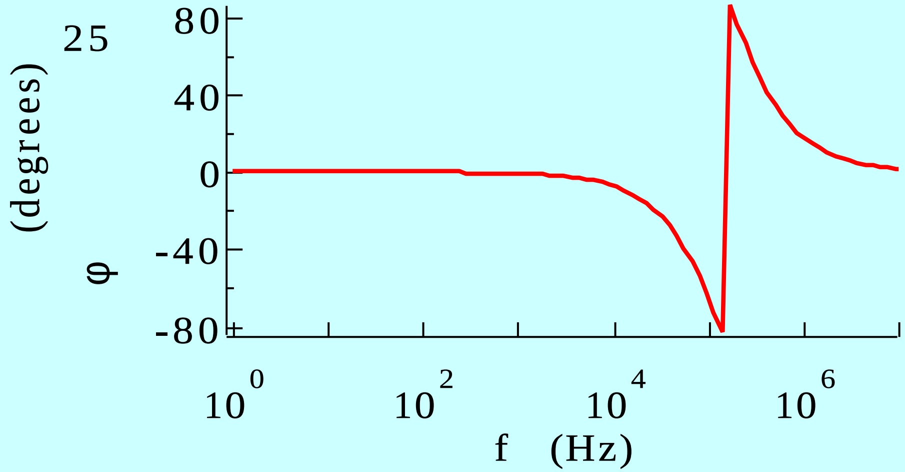
- Commonly encountered pulse shape in amplifier systems

integrator response = $1/j\omega C$

$F(\omega) = A/(1+j\omega\tau)^2$







Differentiation and integration

$$FT[f'(t)] = \int_{-\infty}^{\infty} f'(t).e^{-j\omega t}.dt$$

$$= \int_{-\infty}^{\infty} \lim \left[\frac{f(t+\Delta t) - f(t)}{\Delta t} \right].e^{-j\omega t}.dt$$

limit at Δt -> 0

$$= \int_{-\infty}^{\infty} \lim \left[\frac{f(t+\Delta t)]e^{-j\omega t}}{\Delta t} \right].dt - \int_{-\infty}^{\infty} \lim \left[\frac{f(t)].e^{-j\omega t}}{\Delta t} \right].dt$$

$$= \lim \left[\frac{e^{j\omega\Delta t} F(\omega) - F(\omega)}{\Delta t} \right] = j\omega F(\omega)$$

use Shift theorem

$$FT\left[\int_{-\infty}^{t} f(t)dt\right] = \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{t} f(u)du.\right\} e^{-j\omega t}.dt$$

let $\int_{-\infty}^{t} f(u)du = g(t)$

$$\int_{-\infty}^{\infty} \left\{\int_{-\infty}^{t} f(u)du.\right\} e^{-j\omega t}.dt = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}.dt$$

$$= \left[g(t)\, e^{-j\omega t}/(-j\omega)\right]_{-\infty}^{\infty} + (1/j\omega) \int_{-\infty}^{\infty} g'(t)e^{-j\omega t}.dt$$

$$= F(\omega)/j\omega$$

Formally, subject to constraints on $g(\pm\infty)$

Fourier transforms of repetitive functions

- typically give line spectra, instead of continuous

  ie series of discrete frequency components dominate

  obvious for $\sin(\omega_0 t)$ *and combinations*

- Recall Modulation theorem

  $FT[f(t)\cos\omega_0 t] = (1/2)\{F(\omega-\omega_0) + F(\omega+\omega_0)\}$

  so $f(t) = 1$ $F(\omega) = 2\pi\delta(\omega)$

  $FT[\cos\omega_0 t] = (1/2)\{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\}$

  single freqency component at $\omega = \omega_0$ (and $-\omega = \omega_0$)

  $FT[\cos(\omega_0 t)\cos(\omega_1 t)] =$

  $(1/4)\{\delta(\omega-\omega_0-\omega_1) + \delta(\omega-\omega_0+\omega_1) + \delta(\omega+\omega_0-\omega_1) + \delta(\omega+\omega_0+\omega_1)\}$

  components at $\omega = \omega_0 - \omega_1$ and $\omega = \omega_0 + \omega_1$ (and $-\omega =$ ...)

- What is the meaning of negative frequencies?

Negative frequencies

- Can consider them as a formal mathematical consequence of the Fourier integral which has an elegant symmetry

but doesn't interfere with practical applications

We are always concerned with functions which are real

since measured quantities must be

For real functions $F(-\omega) = F^*(\omega)$

and we always encounter combinations like $\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$

$$\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega = \int_{-\infty}^{0} F(\omega)\, e^{j\omega t} d\omega + \int_{0}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

$$= \int_{\infty}^{0} -F(-\upsilon)\, e^{-j\upsilon t} d\upsilon + \int_{0}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

$$= \int_{0}^{\infty} F^*(\omega)\, e^{-j\omega t} d\omega + \int_{0}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

if $F(\omega) = F_0 e^{j\theta}$

then $F^*(\omega)\, e^{-j\omega t} + F(\omega)\, e^{j\omega t} = F_0[e^{-j(\omega t+\theta)} + e^{j(\omega t+\theta)}]$

so $\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega = 2\int_{0}^{\infty} F_0 \cos(\omega t+\theta)\, d\omega$ purely real integral

Sequence of pulses

- General case

$g(t) = \sum_{n=-\infty}^{\infty} f(t+n\Delta t)$

$\ldots\ f(t+2\Delta t) + f(t+\Delta t) + f(t) + f(t-\Delta t) + f(t-2\Delta t) + \ldots\ f(t-n\Delta t) + \ldots$

from Shift theorem

$G(\omega) = F(\omega) \sum_{n=-\infty}^{\infty} e^{j\omega n\Delta t} = F(\omega)\,[1+ \sum_{n=1}^{\infty} 2\cos(\omega n\Delta t)]$

$\sum_{n=-\infty}^{\infty} e^{j\omega n\Delta t} = \sum_{n=-\infty}^{\infty} e^{j\Theta} = \sum_{n=0}^{\infty} e^{j\Theta} + \sum_{n=0}^{\infty} e^{-j\Theta} - 1$

Geometric series $S = 1 + x + x^2 + x^3 + \ldots x^n + \ldots = 1/(1-x)$

$\sum_{n=-\infty}^{\infty} e^{j\Theta} = 1/(1- e^{j\theta}) + 1/(1- e^{-j\theta}) - 1 = 1$

so $G(\omega) = F(\omega)$

frequency content unchanged - as seems logical

but normally can't observe waveform for infinite time

If f(t) is truly periodic
ie duration < Δt

we'll later find it more convenient to work with Fourier series

exploit the natural harmonics of the system

Real sequences

- If observe for a duration T, the lowest freqency which can be observed is ~1/T

  ie partial cycles should be included with random phase and would be expected not to contribute

- so convolute periodic waveform with top-hat duration T to make it finite

  $g(t) = \sum_{n=-\infty}^{\infty} f(t+n\Delta t) * \Pi(t,T)$

  $G(\omega) = F(\omega).2\sin(\omega T/2)/\omega$

  this has peaks at $\omega T/2 = (\pi/2)(2k+1)$ $k = 1, 2, 3,\ldots$

  ie multiples of $\omega_0 = (\pi/T)(2k+1)$

- Train of rectangular pulses, duration a

  $G(\omega) = [2\sin(\omega a/2)/\omega].\, [2\sin(\omega T/2)/\omega]$

  $= (4/\omega^2)\sin(\omega a/2).\sin(\omega T/2)$

will return to this later