Real sequences

- If observe for a duration T, the lowest freqency which can be observed is ~1/T

  ie partial cycles should be included with random phase and would be expected not to contribute

- so convolute periodic waveform with top-hat duration T to make it finite

  $g(t) = \sum_{n=-\infty}^{\infty} f(t+n\Delta t) * \Pi(t,T)$

  $G(\omega) = F(\omega).2\sin(\omega T/2)/\omega$

  this has peaks at $\omega T/2 = (\pi/2)(2k+1)$ $k = 1, 2, 3,...$

  ie multiples of $\omega_0 = (\pi/T)(2k+1)$

- Train of rectangular pulses, duration a

  $G(\omega) = [2\sin(\omega a/2)/\omega].\ [2\sin(\omega T/2)/\omega]$

  $= (4/\omega^2)\sin(\omega a/2).\sin(\omega T/2)$

will return to this later

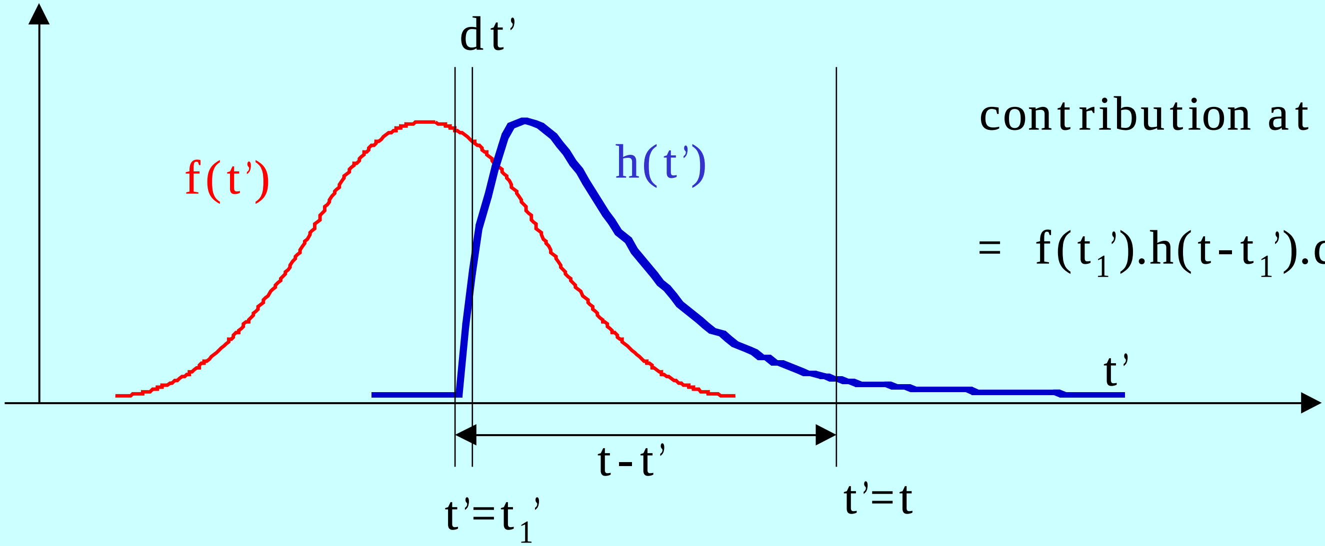
Impulse response and convolution

- generalised multiplication

if a signal f(t) is the input to a system, what is the outcome?

We know the response of the system to an impulse is h(t) ...

ie. impulse at t= 0 gives output h(t) at t



contribution at t from signal at t_1'

$= f(t_1').h(t-t_1').dt'$

Consider signal as made of series of impulses with weight f(t)

then $g(t) = \int_{-\infty}^{t} f(t').h(t-t').dt'$

NB integral extends to $-\infty < t' < t$ only

results can't be influenced by times later than measurement

however general convolution does not have this restriction

Convolution theorem

$h(t) = 0$ for $t < 0$ simple statement of causality

so can extend upper limit of integral to $t' = \infty$ without problem, and

$$g(t) = \int_{-\infty}^{t} f(t').h(t-t').dt' = \int_{-\infty}^{\infty} f(t').h(t-t').dt'$$

Th

(not all functions have this causal constraint so integration to ∞ is normal)

Let's find F. Transform (change t' to u to avoid confusion)

$$G(\omega) = FT[\int_{-\infty}^{\infty} f(u).h(t-u).du\,] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u).h(t-u).du.e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(u)\,\{\int_{-\infty}^{\infty} h(t-u).e^{-j\omega t}dt\}\,.du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-j\omega u}\, H(\omega)\,.du$$

$$= F(\omega)H(\omega)$$

Convolution $= f(t)*g(t) =$ multiplication of FTs

•NB because f <==> FT is symmetric, there is a similar result for $F(\omega)*G(\omega)$

Digression

- It's an interesting fact that complex exponentials are eigenfunctions of a Linear Time Invariant (LTI) system. To see this

$$g(t) = \int_{-\infty}^{\infty} f(u).h(t-u).du = \int_{-\infty}^{\infty} h(u).f(t-u).du$$

to get this, we assumed the system was linear and time invariant

put $f(t) = e^{j\omega t}$

$$g(t) = \int_{-\infty}^{\infty} h(u)e^{j\omega t}e^{-j\omega u}du$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(u).e^{-j\omega u}du$$

$$= H(\omega)\, e^{j\omega t}$$

- This is another argument for the use of such signals in analysing systems

Parseval's (Rayleigh's) & Power theorems

- need this result

if $G(\omega) = \int_{-\infty}^{\infty} g(t).e^{-j\omega t}.dt$

then $G^*(\omega) = \int_{-\infty}^{\infty} g^*(t).e^{j\omega t}.dt$

- wish to find $\int_{-\infty}^{\infty} f(t).g^*(t).dt$

$$\int_{-\infty}^{\infty} f(t).g^*(t).dt = (1/2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega).g^*(t).e^{j\omega t}d\omega.dt$$

$$= (1/2\pi)\int_{-\infty}^{\infty} F(\omega)\,\{\int_{-\infty}^{\infty} g^*(t).e^{j\omega t}dt\}\,.\,d\omega$$

$$= (1/2\pi)\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$$

$$= \int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,df$$

Special case when $g^*(t) = f^*(t)$

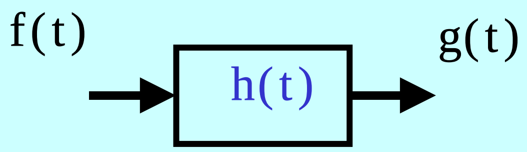
$$\int_{-\infty}^{\infty} |f(t)|^2.dt = \int_{-\infty}^{\infty} |F(\omega)|^2 df$$

Impulse response and transfer function relationship

- Signal processing system , eg. Amplifier

output = convolution of signal and impulse response in time domain

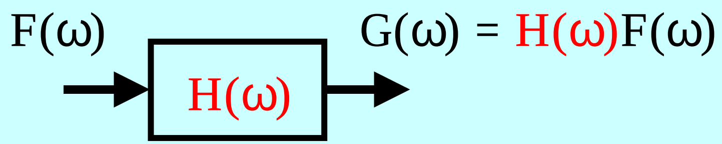
ie $g(t) = f(t)*h(t)$



and from convolution theorem

$G(\omega) = F(\omega)H(\omega)$

where $G(\omega) = FT[g(t)]$



but we already know that the spectral content at the output is the product of the spectral content of the signal and the transfer functions

so the transfer function and impulse response are a Fourier transform pair

Bandwidth and duration

- Equivalent area

$F(\omega) = \int_{-\infty}^{\infty} f(t).e^{-j\omega t}.dt$ so $F(0) = \int_{-\infty}^{\infty} f(t)\,dt$ = area under f(t)

and similarly $f(0) = \int_{-\infty}^{\infty} F(\omega)\,df$

define equivalent area = area under curve/height at [t/ω=] 0

thus $$\frac{\int_{-\infty}^{\infty} f(t)\,dt}{f(0)} = \frac{F(0)}{\int_{-\infty}^{\infty} F(\omega)\,df}$$

ie. reciprocal relation between equivalent area in time and frequency

increase width of one, other decreases

examples $\delta(t)$ <-> 1

$\Pi(t)$ <-> $2\sin(\omega a)/\omega$

$\exp(-a^2t^2)$ <-> $(\sqrt{\pi}/a)\exp(-\omega^2/4a^2)$

convince yourself this is true

- Bandwidth x duration = constant

mathematical consequence of interrelation of f and t

Uncertainty principle

- Define $(\Delta t)^2$ and $(\Delta\omega)^2$ as variances in t and ω

  $\langle x\rangle = \int x.p(x).dx$ $\quad\quad$ $\langle x^2\rangle = \int x^2.p(x).dx$ etc

  there is more than one possible way of calculating these values

  choose appropriate probability distribution p(x) $\quad$ [NB $\int p(x).dx = 1$]

  the choice could be f(t) or F(ω) ***but***

  a useful choice with much practical value is

  $p(t) = ff^*$ or $p(\omega) = FF^*$ (properly normalised)

  then variance is calculated by weighting with Power (Intensity) spectrum

- $$(\Delta t)^2 = \frac{\int_{-\infty}^{\infty} t^2.|f(t)|^2.dt}{\int_{-\infty}^{\infty} |f(t)|^2.dt} \qquad (\Delta\omega)^2 = \frac{\int_{-\infty}^{\infty} \omega^2.|F(\omega)|^2.df}{\int_{-\infty}^{\infty} |F(\omega)|^2.df}$$

  can be shown in very general way that $\Delta t.\Delta\omega \geq 1/2$ or $\Delta t.\Delta\nu \geq 1/4\pi$

  which is often known as the ***Bandwidth Theorem***

  a pulse is said to be **transform limited** if it contains the minimum number frequencies sufficient to support the pulse shape

  it is possible to have more frequencies in pulses, satisfying $\Delta t.\Delta\omega > 1/2$

Small footnote

- Should be well known but...

- mean and σ calculated from probability distribution p(x)

$$\int p(x)\ dx = 1$$

$$\langle x \rangle = \int x.p(x).\ dx$$

$$\langle x^2 \rangle = \int x^2.p(x).dx$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$\sigma^2 = \langle x^2 \rangle$ only when $\langle x \rangle = 0$

so for symmetric distributions like gaussian $\sigma^2 = \langle x^2 \rangle$

Gaussian pulses and uncertainty

- Gaussian pulses transform to gaussian pulses

  $f(t) = \exp(-a^2t^2)$ $\qquad$ $F(\omega) = (\sqrt{\pi}/a)\exp(-\omega^2/4a^2)$

  in optics, laser spatial profiles are often chosen to be gaussian

- The general form of gaussian probability distribution

  $p(x) = [1/(2\pi\sigma^{1/2})]\exp\{-(x-x_0)^2/2\sigma^2\}$

  mean = x_0 $\qquad$ variance = σ^2 $\qquad$ $\int_{-\infty}^{\infty} p(x)dx = 1$

When evaluating σ_t and σ_ω remember that the appropriate gaussian distributions apply to **power** and not amplitude. In quantum mechanics the probability p(x) = $|\psi(x)|^2$ so the results are identical.

Can show that gaussian pulses satisfy this bound exactly.

$\sigma_t.\,\sigma_\omega = 1/2$ $\qquad$ (on problem sheet)

In optics experiments, this could be used as a useful **reality check** on a super-fast optical pulse experimental measuring both σ_t and σ_ω

Most (all?) other pulse shapes have $\qquad$ $\sigma_t.\,\sigma_\omega > 1/2$

Ultimate bandwidth limitation

- In optical systems often assume that transmitter is very broad-band source

  ie spectral linewidth large compared to modulation bandwidth of signal

  constant pressure to push to the limits for many applications

  gives an interesting example of ...

- Ultimate limit from Fourier transform & uncertainty principle

  the shorter the pulse, the broader the spectrum

  more rapidly degraded by chromatic dispersion

- A communications system wants to send pulses long distances by optical fibre

  a gaussian pulse shape is chosen

  the initial spread in the pulse is $\sigma_0(t)$

  after a distance length L, at wavelength λ

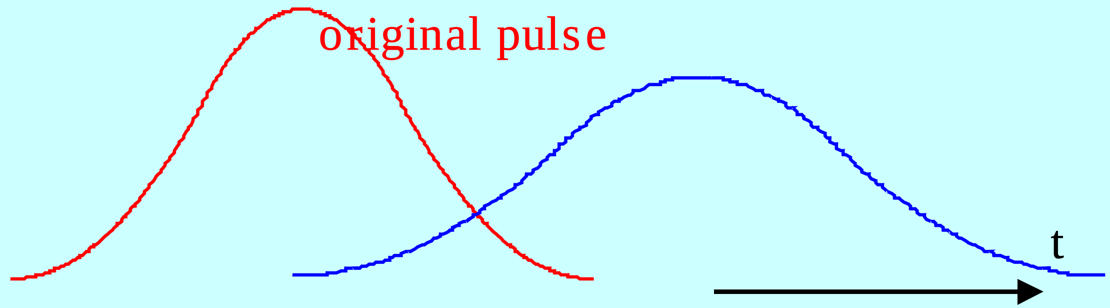
  the result of dispersion is a broadening of the pulse

  $$\sigma^2(t) = \sigma_0^2 + \sigma_D^2 = \sigma_0^2 + D_m^2\sigma_\lambda^2L^2$$

- what is the best value of $\sigma(t)$ and the speed of optical transmission?

Dispersion and bandwidth

- $\sigma^2(t) = \sigma_0{}^2 + \sigma_D{}^2 = \sigma_0{}^2 + D_m{}^2\sigma_\lambda{}^2L^2$



single mode fibre

and $\lambda = 1550$nm

$L = 100$km

measured dispersion $D_m = 15$ps/km.nm

different spectral components travel at slightly different speeds

$\sigma_\lambda{}^2 = 4\pi^2c^2\sigma_\omega{}^2/\omega^4 = \sigma_\omega{}^2\lambda^4/4\pi^2c^2 = \lambda^4/16\pi^2c^2\sigma_0{}^2$

since $\sigma_t = 1/2\sigma_\omega$ for gaussian

$\sigma^2 = \sigma_0{}^2 + D_m{}^2\sigma_\lambda{}^2L^2 = \sigma_0{}^2 + A^2/\sigma_0{}^2$ $\qquad A = D_mL\lambda^2/4\pi c$

Minimum is when $\sigma_0{}^4 = A^2$ so $\sigma^2 = 2\sigma_0{}^2$

$\sigma_{min} = \lambda(D_mL/2\pi c)^{1/2} = 44$ ps

ie. starting with shorter pulse will lead to more dispersion and longer pulse at receiver

- Data transmission rate?

Maximum bit rate

•How closely separated can two pulses be in time?

envelope is

$f(t) = \exp\{-t^2/2\sigma^2\} + \exp\{-(t-t_0)^2/2\sigma^2\}$

could find general solution by minimising

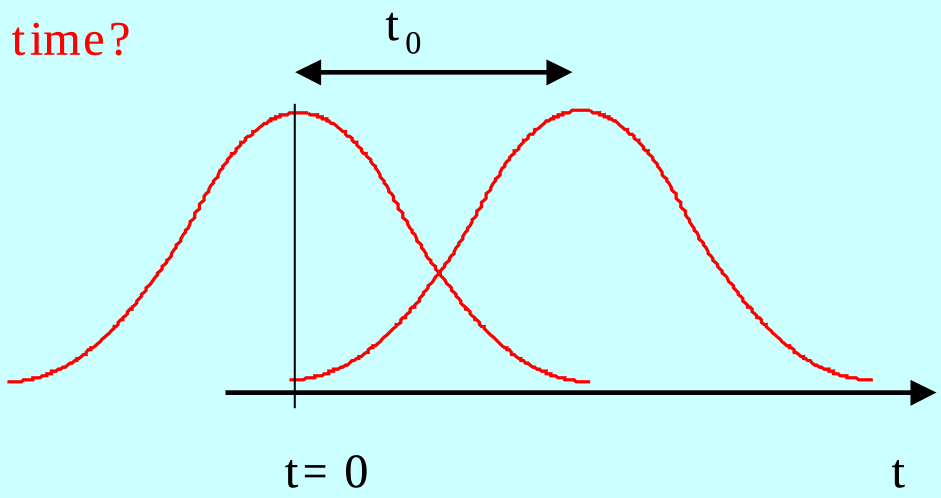
complicated!

but <u>usually</u> a minimum at $t = t_0/2$

$f(t_0/2) = 2\exp\{-t_0^2/8\sigma^2\}$

while $f(0) = f(t_0)$ is <u>usually</u> a maximum

$f(0) = 1 + \exp\{-t_0^2/2\sigma^2\}$



t_0	$f(0) = 1+\exp(-t_0^2/2\sigma^2)$	$f(t_0/2) = 2\exp(-t_0^2/8\sigma^2)$
σ	1.61	1.77
2σ	1.14	1.21
3σ	1.01	0.65
4σ	1.00	0.27
5σ	1.00	0.09

•good separation at $t_0 \approx 4\sigma$

so maximum bit rate

is $\approx 1/4\sigma \approx 5.7$ Gb/s

•I've considered amplitudes - should consider power?

•Could we do better with any other pulse shape?

Power spectral density

•Many functions we are dealing with represent

f(t) = voltage or current

f(t) = amplitude (eg of light pulse)

•In such cases, the total energy or intensity is

$\Delta E = \int_{t1}^{t2} |f(t)|^2 dt$ energy delivered in interval $t_1 < t \leq t_2$

or, in frequency interval,

$\Delta E = \int_{f1}^{f2} |F(\omega)|^2 df$ energy in range $f_1 < f \leq f_2$

with an appropriate factor of R, for V & I

•Power spectral density $W(\omega) = |F(\omega)|^2$

remembering the integration is in f, not ω

otherwise need a (1/2π) factor

Bandpass filters

- We will encounter many systems where we are interested in estimating the bandwidth

ie the range of frequencies transmitted by the system

In ideal cases we would often like to simplify this by assuming that all frequencies in a range are transmitted without attenuation

ie $H(\omega) = 1$ for $\omega_1 < \omega < \omega_2$

We can now see that this simple picture is physically impossible to realise since it would imply

infinite range of frequencies

an impulse response of $h(t) \approx e^{-j(\omega 1+\omega 2)t}. [2\sin(\omega_1+\omega_2)t/2]/\pi t$

(Symmetry and shift theorems)

complex and oscillatory - not practical to realise

however, this does not stop us using the concept

nor defining **effective bandwidth**