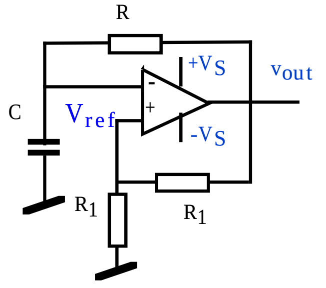
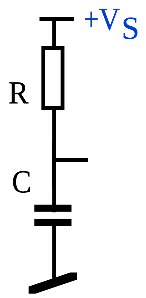
Instrumentation Problem Sheet 5 Answers

(1) This oscillator works by charging and discharging the capacitor to a voltage which triggers the op-amp, or comparator. When the capacitor voltage crosses the threshold, the output swings rapidly between the supply voltages. The positive feedback across R_1 enhances this, and this pair of resistors also sets the threshold level. Start by assuming v_{out} is high, ie $v_{out} = +V_s$. Then $V_{ref} = +0.5V_S$. If the output is low, $V_{ref} = -0.5V_S$. Thus the capacitor voltage must be charging and discharging between $-0.5V_S$ and $+0.5V_S$.



If V_C is somewhere in this range and v_{out} is low, a current will flow discharging C. It discharges until $V_C < V_{ref}$, at which point $v_- < V_{ref}$ so the comparator switches state to the high level, and so on. We need to find how long it takes the capacitor to charge from $-0.5V_S$ to $+0.5V_S$.
The condition for the current to flow through resistor R is shown in the next figure:



The equation is $V_S = IR + Q/C = RdQ/dt + Q/C$ whose solution is

$$CV_S(e^{t/RC}-1) = Q(t)e^{t/RC} - Q(0)$$

If the capacitor is charging from the positive rail, then $Q(0) = -0.5CV_S$ so

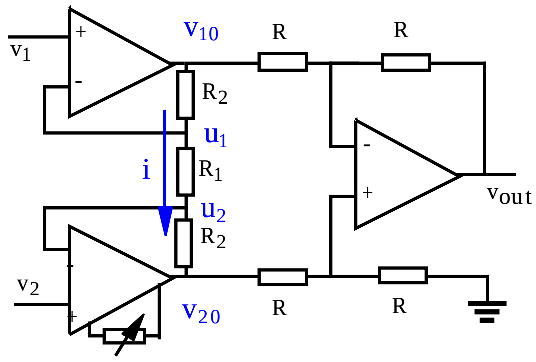
$$Q(t) = CV_S[1-(3/2)e^{-t/RC}]$$

and charging continues until time t when $V(t) = +0.5V_S$. This occurs when $t = RC\ln(3)$, so the period is twice this, which is

$$T = 2RC\ln(3) \approx 2.2RC$$

The waveform at the inverting input of the op-amp is a sawtooth shape, made up of the rising and falling exponentials.

(2) The offset null allows to correct for voltage offsets at the output originating in either small currents flowing into the op-amp or slightly unbalanced voltages at the input arising from manufacturing variations construction of the circuit. Typically a potentiometer is connected between two pins of the circuit and can be fine tuned to zero the output.



Referring to the figure

$u_1 = v_1$ and $u_2 = v_2$ (using $v_- = v_+$)

$i = (u_1 - u_2)/R_1 = (v_{10} - u_1)/R_2 = (u_2 - v_{20})/R_2$ (no current flow into op-amp)

adding and multiplying by R_2

$$(v_{10} - v_{20}) - (u_1 - u_2) = 2iR_2$$

then substituting for i again

$$(v_{10} - v_{20}) = (u_1 - u_2)(1 + 2R_2/R_1) = (v_1 - v_2)(1 + 2R_2/R_1)$$

so $G_{diff} = (1 + 2R_2/R_1)$

For the common mode gain, put $v_1 = w_1 + v_{CM}$ and $v_2 = w_2 + v_{CM}$
then $(v_{10} + v_{20}) = \quad (v_1 + v_2)$
and if $w_2 = w_2 = 0$ then $v_1 + v_2 = 2v_{CM}$ and $v_{10} = v_{20}$ so $G_{CM} = 1$

At the second stage we have (the differential amplifier of Problem Sheet 4)

$$(v_{10} - v_{out})/2 + v_{out} = v_- \qquad \text{and} \qquad v_{20}/2 = v_+$$

so, with not much algebra $v_{out} = (v_{20} - v_{10})$ so $G_{diff2} = 1$
For the common mode, $v_{out} = 0$ if $v_{10} = v_{20} = v_{CM}$ so $G_{CM2} = 0$

(3) To start with, ignore the negative feedback loop. The non-inverting feedback loop has two components (Z_1 with R and C in parallel and Z_2 with R and C in series)

$$Z_1 = R/(1+j\omega\tau) \quad Z_2 = (1+j\omega\tau)/j\omega C \quad \text{with} \quad \tau = RC$$

Thus $G = v_{out}/v_+ = 1 + Z_2/Z_1 = 1 + (1+j\omega\tau)^2/j\omega\tau \quad = 1 + (1 - \omega^2\tau^2 + 2j\omega\tau)/j\omega\tau$

$$= 1 + (1 - \omega^2\tau^2 + 2j\omega\tau)/j\omega\tau = 3 - j(1 - \omega^2\tau^2)/\omega\tau$$

$$v_+ = v_{out}/G$$

At $\omega\tau = 1$, the positive input has a phase shift = 0, so the circuit has <u>positive</u> feedback at this frequency with 1/3 of the output voltage passed back to the input. This would lead to instability and the circuit would rapidly go into saturation were it not for the negative feedback path. If the same fraction of the output is passed back to the negative input the two inputs would cancel, so the circuit is tuned so that the negative feedback almost, but not quite, matches the positive feedback. The circuit will naturally oscillate but the negative feedback prevents complete instability by controlling the amplitude of the oscillation.
If the lamp is rated at 6V, 40mA it has a resistance of approximately 150Ω at the operating temperature, so the feedback resistor r should be chosen to have a value of about 300Ω. In practice, it is convenient to have a small adjustable component so the value can be set. As the oscillations grow, the lamp becomes hotter, increasing its resistance and increasing the feedback fraction. So the stability of the circuit is maintained.
For RC = 100kΩx1.5nF = 0.15ms, f_0 = 1.06kHz. At higher frequencies (shorter RC) the phase shift of G at frequencies near to f_0 will become important. For frequencies in the audio range (~10Hz – 20kHz), the op-amp is operating where its gain is close to the DC value and the phase shift is very small (remember the Bode plot of the op-amp). At higher frequencies, the phase shift grows so that positive feedback occurs at frequencies away from the chosen operating point, causing distortion. For audio applications, eg component evaluation, a circuit with a very pure oscillation is preferred.

(4) This is certainly a transmission line. The speed of transmission on the cable and its characteristic impedance are

$$v = 1/(LC)^{1/2} = 1.4\ 10^8\ \text{m.s}^{-1} \approx 0.5c \quad \text{and } Z_0 = (L/C)^{1/2} \approx 70\Omega$$

At the high frequencies contributing to such a fast pulse the wavelengths are much shorter than 10m, so the impedance is 70Ω, independent of length.
If the capacitance were 10pF.m^{-1} then

$$v = 1/(LC)^{1/2} = 4.5\ 10^8\ \text{m.s}^{-1} \approx 1.5c \quad \text{so is not physically possible.}$$

The cable is not terminated in its characteristic impedance, therefore there will be reflections. The reflection coefficients are

$R_o = (100-70)/(100+70) = 0.18$ and $R_s = (10-70)/(10+70) = -0.75$

The signal is not inverted on its first reflection ($R_0 > 0$) and it has an amplitude of 1.8V. The transmission time down 200m is 1.4µs so the first reflection arrives at the output 4.2µs after the first pulse was injected into the cable. It has an amplitude of

10Vx0.18x-0.75 = -1.35V ie inverted (at the sending end) with respect to the initial pulse. After a further round trip which arrives at the output at 7.0µs, the reflected signal has an amplitude

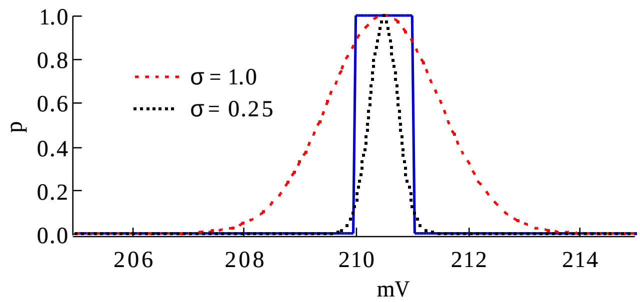
-1.35Vx0.18x-0.75 = 0.18V ie the same polarity as the initial pulse,

Because the scope has a high impedance, which is in parallel with the load, it does not change the termination properties. The pulse which is observed is the sum of the incident and reflected pulses, or $V = V_{input}T = V_{arriving}x2Z_L/(Z_L+Z_0) = 1.18V_{arriving}$.

Thus, at 1.4µs, a 11.8V pulse is observed, at 4.2µs a -1.59V pulse is observed, and at 7.0µs a 0.21V pulse is observed.

(5) A 10-bit ADC has 2^{10} = 1024 divisions, so the Least Significant Bit (LSB), or one bin, represents a value of 1V/1024 ≈ 1mV. If the comparator threshold has 1mV rms noise, we would expect that the voltage which triggered the ADC, for some particular level, to fluctuate with a gaussian distribution centred on the level with σ = 1mV. If that were so, the values supposed to be contained in one bin of the ADC would be spread over several bins.

Thus, the rms noise must be several times smaller. A reasonable value might be 0.25mV. The figure shows one (arbitrary) bin of a 10bit ADC with two gaussian distributions superimposed.



(6) The precision on the time is given by

$\sigma(t) = \sigma(v)/(dV/dt) = \sigma(v)/a = 80mV/(40mV/ns) = 2ns$

The precision on the timing should not depend on the value of the threshold if the rise time is linear, as stated in the problem. All that would change is the delay in the generation of the timing signal. Eg if the threshold were changed from 200mV to 400mV, the timing signal would be generated 5ns later.

In reality such linearity would not be expected over the full range of signals. More likely this is an approximation to a voltage which would be more exponential in shape, ie $v(t) = V_{max}[1-\exp(-t/\tau)]$. In that case, setting the threshold too high would worsen the timing precision, while setting it too low would mean more frequent noise triggers. As usual, life is a compromise!

(7) The count rate is given by the level crossing rate with a threshold V, which is

$f(V) = f(Z).\exp(-V^2/2\sigma^2)$

where f(Z) is the positive zero crossing rate (ie in one direction only).

So $f(25)/f(10) = \exp(-25^2/2\sigma^2)/\exp(-10^2/2\sigma^2)$ with σ = 5.

$= \exp(-12.5)/\exp(-2) = 2.8\ 10^{-3}$

so the count rate should fall from 200kHz to about 5Hz. If it did not, it would be wise to check that the noise has a gaussian amplitude distribution. If it did not, then start looking for extra noise in the system, eg from interference or poor grounding.

(8) Because this is a current amplifier, it is most convenient to work in terms of currents and current noise. If the transistor has $\beta = 100$ and $I_C = 0.4$mA, then $I_B = I_C/100 = 4\mu$A. This is far larger than the detector leakage current so will dominate the shot noise. The feedback resistor is also a parallel noise source. It can be compared with the transistor input current by equating the spectral noise density of the resistor to the shot noise produced by a current I_{eq}, ie

$$4kT/R = 2eI_{eq} \quad \text{or} \quad I_{eq} = (2kT/e)(1/R)$$

At room temperature $kT/e \approx 25$mV so $I_{eq} \approx 50\mu A/k\Omega$. So the 10kΩ resistor is equivalent to 5μA. The total equivalent input current for noise calculations is then

$$I = (5\mu A^2 + 4\mu A^2)^{1/2} \approx 6.4\mu A.$$

The total noise at the input, expressed as a current, is $I_n^2 = e_n^2\omega^2C^2 + i_n^2$ and the corner frequency is when these two terms are equal in magnitude, so

$$\omega^2 = i_n^2/e_n^2C^2 \quad \text{with} \quad e_n^2 = (2e/I_C)(kT/e)^2 \quad \text{and} \quad i_n^2 = 2eI_{eq}$$

With the values given:

$$e_n^2 = 5.0\text{x}10^{-19}\ V^2/Hz \qquad i_n^2 = 2.0\text{x}10^{-24} A^2/Hz$$

$$\omega_{corner} = 10^8 \text{ rad/s} \qquad \tau_{corner} = 10\text{ns} \qquad f_{corner} = 16\text{MHz}$$

The optimum filter would therefore have a time constant ≈ 10ns.

The noise can be expressed as

$$ENC^2 = e_n^2C^2\int[h'(t)]^2.dt/2 + i_n^2\int[h(t)]^2.dt/2$$

where h(t) is the impulse response. which should be normalised so that $h_{max}(t) = 1$. The factors of (1/2) were explained in the lectures.

Thus

$$\begin{array}{llll} h(t) = 1 - t/T & h'(t) = -1/T & 0 < t < T & \text{where } T = 15\text{ns} \\ \quad\;\; = 1 + t/T & h'(t) = 1/T & -T < t < 0 & \\ \quad\;\; = 0 & \text{elsewhere} & & \end{array}$$

so $\int[h(t)]^2.dt/2 = T/3$ and $\int[h'(t)]^2.dt/2 = 1/T$

With the values given

$$ENC \approx 1.5\text{x}10^{-16} \text{ coulombs} \quad \approx 960 \text{ electrons}$$

(9) Digital data transmission has advantages of noise immunity because the only requirement is to distinguish a LOW (0) from a HIGH (1) level. Binary signals can be transmitted over long distances at multi-Gb/s rates using optical fibres and even higher rates are possible over short distances. In long distance communications copper cables would have too much attenuation to support such rates and even fibre attenuation requires repeater amplifiers over very long distances. Nevertheless binary signals can be restored without loss of information.

However, it also means that signals to be transmitted must be digitised and high resolution digitisation is expensive, power hungry and difficult at high speed. Bit errors (ie 0 or 1 mistransmitted) do occur and can have significant effects – eg a high bit of a signal value or a control word. Each bit requires a clock cycle to transmit so, in some cases, analogue information can offer comparable rates. Digital transmission is well matched to computer networks, where signals are already binary, and speech transmission, where the bandwidth in a single conversation is much smaller than the bandwidth of the data links.

Analogue data transmission avoids the need to digitise signals which eliminates some elements of power consumption. This can be a very important consideration in some applications, but it does require that the transmitter should be sufficiently linear to support the range of amplitudes of the signals. Digital data also require coding and encoding electronics to protect against occasional errors, which add complexity to the system. High bandwidth analogue transmission is used in cable TV distribution where a single signal can be transmitted to many destinations. As yet, digital TV and video is still developing and very likely will dominate this area in the near future.

The problem of chromatic dispersion in an optical fibre was discussed in Lecture 4, on Fourier transforms, in conjunction with the Uncertainty Principle. If the pulse is gaussian, it satisfies the limit that $\sigma(t)\sigma(\omega) = 1/2$.

Since $\lambda\omega = 2\pi c$, then $\delta\lambda = -2\pi c\delta\omega/\omega^2$ or $\sigma(\lambda) = (2\pi c\omega^{-2})\sigma(\omega)$. Here c is c_{vacuum}/n where n is the refractive index (I think this factor was omitted from the calculation in the lecture notes.)

Notes: $\sigma^2(L) = \sigma^2(L{=}0) + D^2\sigma^2(\lambda)L^2 = \sigma_0{}^2 + D^2\sigma^2(\lambda)L^2$

The equation is from optical communications and it expresses the relationship between the final width of the pulse and the initial values. Don't interpret D and $\sigma^2(\lambda)$ as continuously varying! D should be assumed to be constant at a given operating wavelength - a large change in wavelength, eg from 1550nm to 1310nm, is required to change it significantly.

$$\sigma^2 = \sigma_0{}^2 + D^2\sigma_\lambda{}^2L^2 = \sigma_0{}^2 + A^2/\sigma_0{}^2 \qquad A = DL\lambda^2/4\pi c$$

and we can minimise with respect to σ_0, which leads to

$$\sigma_{min} = \lambda(DL/2\pi c)^{1/2}$$

Putting in numbers $c = 2.0\text{x}10^8$ m/s $= 2.0\text{x}10^5$ nm/ps D = 17.5ps/km.nm

σ_{min} = 18.0ps and maximum bit rate = 13.8 Gb/s $=1/4\sigma_{min}$, as explained in the lecture,

The data transmission rate could be improved by operating at a wavelength closer to the dispersion zero, around 1310nm. This has a price, which is that attenuation at shorter wavelengths is larger by about 0.2dB.km, as discussed in another lecture, so more power would be required. This is a typical situation in systems where a decision about which factor is most important must be made.

(10) The statistical fluctuations are Poisson so a measurement of N has a variance σ^2 = N. Therefore, 1% resolution can only be achieved if spots contain $N > 10^4$, then $\sigma(N)/N \leq 0.01$. In that case, the dynamic range of the system is

$$DR = 6.55\text{x}10^8/10^4 = 65{,}500 \approx 2^{16} \text{ (ie 16 bits) or } 96\text{dB}$$

The largest signals have $\sigma(N)/N$ = 0.004%

To fit the range of the ADC, the largest signals should have a 1V amplitude, so

$$G = 1.5\ 10^{-9}\ \text{V/photon}$$

but this means the smallest spots of 10^4 photons will give rise to voltages of 15μV, which is well below the LSB of the ADC (see previous problem) which has a bin size of

1mV. The second gain should therefore be sufficient to raise the signal above 1mV, ie the minimum gain is

$G_2 = 10^{-7}$ V/photon

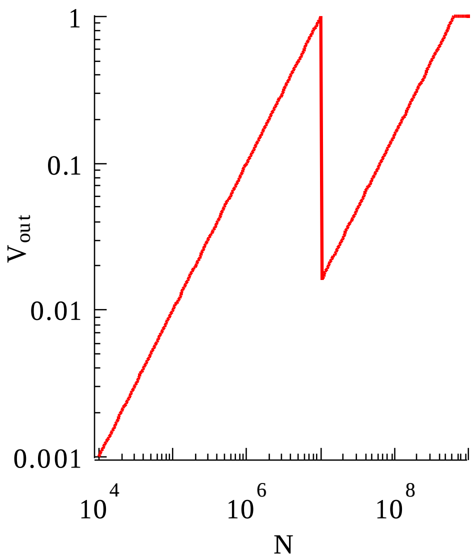
This will saturate the ADC for signals of 10^7 photons. For signals larger than this the lower gain setting is used.

[To be a little more precise, the resolution of the ADC should be taken into account. This is σ = 1mV/√12 = 0.29V, and this should be added in quadrature with the statistical error to calculate the resolution, which affects mainly the smallest signals.]

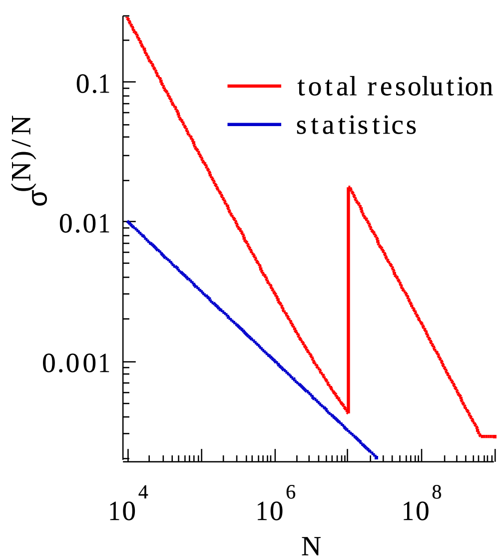
Is 1% resolution achieved over the full dynamic range? We should look carefully at the small signal and gain transition regions, ie signals just larger than 10^7 photons. There, the resolution is dominated by the ADC.

10^7 photons gives a signal of 0.015V. At this point, σ(V)/V = 1.9%

The results are shown in the figure below. If G_2 was increased further, the resolution could be improved in the transition region. This is a large dynamic range to be covered with just two gain settings. At this point, I would be discussing with the experimenters how important the small signal resolution actually is!



Signal size vs number of photons



Resolution vs N