Instrumentation Problem Sheet 2: Fourier transforms

(1) Prove the following:

(i) $FT[f(-t)] = F(-\omega)$

(ii) $FT[f(t).\sin(\alpha t)] = (1/2j)[F(\omega-\alpha) - F(\omega+\alpha)]$

(iii) $FT[d^n f(t)/dt^n] = (j\omega)^n F(\omega)$

(iv) if f(t) is a real function, $F(\omega) = F^*(-\omega)$

(2) Show that the FT of the gaussian function $f(t) = \exp(-a^2t^2)$ is
$F(\omega) = (\sqrt{\pi}/a)\exp(-\omega^2/4a^2)$

Hint: make the exponent in the integrand a perfect square $(at+\alpha)^2$

(3) Show that $FT[\cos(\omega_0 t)] = \pi [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

(4) Show that the FT of the exponential function $f(t) = e^{-at}$ $t\geq 0$ can also be written
$F(\omega) = (a - j\omega)/(a^2+\omega^2)$
What is the phase shift of a filter which has this response?

(5) Show that the FT of the top-hat function $f(t) = 1$ $-a < t \leq a$, $f(t) = 0$ elsewhere is
$F(\omega) = 2\sin(\omega a)/\omega$

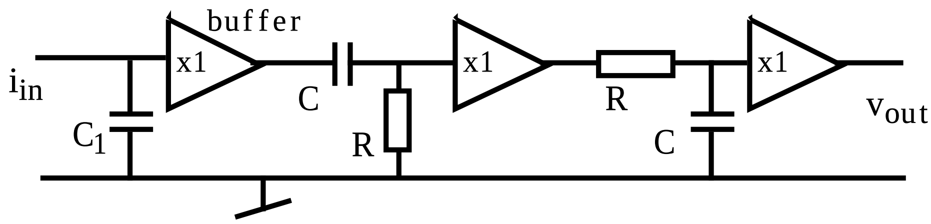
(6) Verify by direct integration that the area under the function f(t) is equal to the FT at $\omega = 0$, $F(\omega)$ for (i) $f(t) = ate^{-at}$, (ii) $f(t) = e^{-at}$. In both cases $f(t) = 0$ for $t < 0$.

(7) Verify that the gaussian pulse shape and its FT exactly satisfy the Uncertainty Principle that $\sigma_t.\sigma_\omega = 1/2$. (NB remember that the bandwidth theorem applies to power spectra, or to probabilities in quantum mechanics).

(8) Using the circuit diagram, write down the transfer function of a low pass RC filter. What is the impulse response?

(9) Using the result of the previous problem, calculate by direct integration the response of an RC low pass filter to a step pulse of duration T. Compare your result to the answer from Problem sheet 1.

(10)



By writing down the transfer function of the system in the figure, and comparing the FT, show that the response of the system to a current impulse is Ate^{-at}. What are the values of A and a? [The “buffer” stages represent high input impedance and low output impedance amplifiers with unity voltage gain.]

Note: For problems(3) and(7) use the standard integrals
$\int_{-\infty}^{\infty}\exp(-x^2)dx = \sqrt{\pi}$ and $\int_{-\infty}^{\infty}x^2\exp(-x^2)dx = (1/2)\sqrt{\pi}$