Inclusion of MEC in the SuSA-based calculations: Status and perspectives

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MEC in QE neutrino scattering

Collaborators:

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**MEC in QE neutrino scattering**

Results from the papers:

1. Meson-exchange currents and quasielastic *Neutrino* cross sections in the superscaling approximation model.

2. Meson-Exchange Currents and Quasielastic *Antineutrino* Cross Sections in the Superscaling Approximation

   I. Ruiz Simo, C. Albertus, J.E. Amaro, M.B. Barbaro, J.A. Caballero, T.W. Donnelly
   *arXiv:1405.4280 [nucl-th]*
Outline

1. Formalism of \((e, e')\) and \((\nu_l, l^-)\)
   - The relativistic Fermi gas
   - Super-scaling approach (SuSA)
   - 2p-2h Meson exchange currents

2. Status of MEC in SuSA

3. Perspectives: new approach to 2p-2h
   - Phase space function
   - Relativistic problems at high \(q\)
   - Angular distribution: new integration method
   - Properties of the 2p-2h phase space
   - The frozen approximation
   - Application to \(\nu\) reactions
1 General formalism

\[ \text{QE} (e, e') \]

\[ \text{QE} (\nu, l^-) \]

\[ W^+ \]

\[ \nu_\mu \]

\[ p \]

\[ n \]
Kinematics

\[ k'_{\mu} = (\epsilon', \vec{k}') \]

\[ k_{\mu} = (\epsilon, \vec{k}) \]

\[ Q_{\mu} = (\omega, \vec{q}) \]

\[ P'_{\mu} = (E', p') \]

\[ P_{\mu} = (E, p) \]

\[ Q^2 = \omega^2 - q^2 < 0 \]
(e, e') formalism

\[
\frac{d\sigma}{de'd\Omega'} = \sigma_{Mott} (\nu_L R_L + \nu_T R_T)
\]

Electron kinematical factors

\[\nu_L = \rho^2, \quad \nu_T = \frac{1}{2}\rho + \tan^2 \frac{\theta}{2}, \quad \rho \equiv \frac{|Q^2|}{q^2}\]

Response functions:

\[R_L = W^{00}\]
\[R_T = W^{11} + W^{22}\]

Hadronic tensor for (e, e')

\[W^{\mu\nu}(q, \omega) = \sum_{fi} \delta(E_f - E_i - \omega) \langle f | J^\mu(Q) | i \rangle^* \langle f | J^\nu(Q) | i \rangle\]

\(J^\mu(Q)\) is the electromagnetic nuclear current
\begin{align*}
\text{(ν}_l, l^- \text{)} \quad \text{formalism} \\

\text{Cross section:} & \quad \frac{d\sigma}{d\Omega'\,d\epsilon'} = \sigma_0 F_+^2 \\
\text{Similar to } \sigma_{\text{Mott}}: & \quad \sigma_0 = \frac{G^2 \cos^2 \theta_c}{2\pi^2} k' \epsilon' \cos^2 \frac{\tilde{\theta}}{2} \\
\text{Fermi constant:} & \quad G = 1.166 \times 10^{-11} \text{ MeV}^{-2} \\
\text{Cabibbo angle:} & \quad \cos \theta_c = 0.975 \\
\text{Generalized scattering angle:} & \quad \tan^2 \frac{\tilde{\theta}}{2} = \frac{|Q^2|}{(\epsilon + \epsilon')^2 - q^2}
\end{align*}
\((\nu_l, l^-)\) formalism (II)

Nuclear structure information:

\[ \mathcal{F}_+^2 = \hat{V}_{CC} R_{CC} + 2\hat{V}_{CL} R_{CL} + \hat{V}_{LL} R_{LL} + \hat{V}_T R_T + 2\hat{V}_{T'} R_{T'} \]

Kinematical factors \(\hat{V}_K\) from the leptonic tensor

\[
\begin{align*}
\hat{V}_{CC} & = 1 - \delta^2 \tan^2 \tilde{\theta} \\
\hat{V}_{CL} & = \frac{\omega}{q} + \frac{\delta^2}{\rho'} \tan^2 \tilde{\theta} \\
\hat{V}_{LL} & = \frac{\omega^2}{q^2} + \left(1 + \frac{2\omega}{q}\frac{1}{\rho'} + \rho\delta^2\right) \delta^2 \tan^2 \tilde{\theta} \\
\hat{V}_T & = \tan^2 \frac{\tilde{\theta}}{2} + \frac{\rho}{2} - \frac{\delta^2}{\rho'} \left(\frac{\omega}{q} + \frac{1}{2}\rho\rho'\delta^2\right) \tan^2 \frac{\tilde{\theta}}{2} \\
\hat{V}_{T'} & = \frac{1}{\rho'} \left(1 - \frac{\omega\rho'}{q}\delta^2\right) \tan^2 \frac{\tilde{\theta}}{2}
\end{align*}
\]

Adimensional variables:

\[
\begin{align*}
\delta & = \frac{m'}{\sqrt{|Q^2|}} \\
\rho & = \frac{|Q^2|}{q^2} \\
\rho' & = \frac{q}{\epsilon + \epsilon'}
\end{align*}
\]
\((\nu_l, l^-)\) formalism (III)

\[
\begin{align*}
R_{CC} &= W^{00} \\
R_{CL} &= -\frac{1}{2} (W^{03} + W^{30}) \\
R_{LL} &= W^{33} \\
R_T &= W^{11} + W^{22} \\
R_T' &= -\frac{i}{2} (W^{12} - W^{21})
\end{align*}
\]

Weak CC hadronic tensor:

\[
W^{\mu\nu}(q, \omega) = \sum_{f_i} \delta(E_f - E_i - \omega) \langle f | J^\mu(Q) | i \rangle^* \langle f | J^\nu(Q) | i \rangle.
\]
Single-nucleon current

Electromagnetic current

\[ j^\mu(p', p) = \bar{u}(p') \left[ F_1 \gamma^\mu + i \frac{F_2}{2m_N} \sigma^{\mu\nu} Q_\nu \right] u(p) \]

Weak CC current \( j^\mu = j^\mu_V - j^\mu_A \).

\[ j^\mu_V(p', p) = \bar{u}(p') \left[ 2F_1^V \gamma^\mu + i \frac{F_2^V}{m_N} \sigma^{\mu\nu} Q_\nu \right] u(p) \]

\[ j^\mu_A(p', p) = \bar{u}(p') \left[ G_A \gamma^\mu + G_P \frac{Q^\mu}{2m_N} \right] \gamma^5 u(p) \]
The relativistic Fermi gas (RFG)

Nuclear response functions for $(\nu_\mu, \mu^-)$ reactions

$$R_K = N \Lambda_0 U_K f_{RFG}(\psi), \quad K = CC, CL, LL, T, T',$$

- $N$ is the neutron number,

- $\Lambda_0 = \frac{\xi_F}{m_N \eta_F^3 \kappa}$, \quad $\eta_F = k_F/m_N$, \quad $\xi_F = \sqrt{1 + \eta_F^2} - 1$.

- Scaling function $f_{RFG}(\psi) = \frac{3}{4} (1 - \psi^2) \theta(1 - \psi^2)$

- Scaling variable

$$\psi = \frac{1}{\sqrt{\xi_F}} \frac{\lambda - \tau}{\sqrt{(1 + \lambda)\tau + \kappa \sqrt{\tau(1 + \tau)}}}$$

- single-nucleon responses $U_K$
Single-nucleon responses, $K = CC$

\[
U_{CC} = U_{CC}^V + (U_{CC}^A)_c + (U_{CC}^A)_{n.c.}
\]

\[
U_{CC}^V = \frac{\kappa^2}{\tau} \left[ (2G_E^V)^2 + \frac{(2G_E^V)^2 + \tau (2G_M^V)^2}{1 + \tau} \Delta \right],
\]

\[
\Delta = \frac{\tau}{\kappa^2 \xi_F (1 - \psi^2)} \left[ \kappa \sqrt{1 + \frac{1}{\tau}} + \frac{\xi_F}{3} (1 - \psi^2) \right]
\]

The axial-vector response is the sum of conserved (c.) plus non conserved (n.c.) parts,

\[
(U_{CC}^A)_c = \frac{\kappa^2}{\tau} G_A^2 \Delta, \quad (U_{CC}^A)_{n.c.} = \frac{\lambda^2}{\tau} G_A^2.
\]
Single-nucleon responses, $K = CL, LL$

\begin{align*}
U_{CL} &= U^V_{CL} + (U^A_{CL})_c. + (U^A_{CL})_{n.c.} \\
U_{LL} &= U^V_{LL} + (U^A_{LL})_c. + (U^A_{LL})_{n.c.},
\end{align*}

The vector and conserved axial-vector parts are determined by current conservation

\begin{align*}
U^V_{CL} &= -\frac{\lambda}{\kappa} U^V_{CC} \\
U^V_{LL} &= \frac{\lambda^2}{\kappa^2} U^V_{CC}
\end{align*}

\begin{align*}
(U^A_{CL})_c. &= -\frac{\lambda}{\kappa} (U^A_{CC})_c. \\
(U^A_{LL})_c. &= \frac{\lambda^2}{\kappa^2} (U^A_{CC})_c.
\end{align*}

Non-conserved n.c. parts:

\begin{align*}
(U^A_{CL})_{n.c.} &= -\frac{\lambda \kappa}{\tau} G'^2_A \\
(U^A_{LL})_{n.c.} &= \frac{\kappa^2}{\tau} G'^2_A.
\end{align*}
Single-nucleon responses, $K = T, T'$

\[
U_T = U_T^V + U_T^A
\]

\[
U_T^V = 2\tau (2G_M^V)^2 + \frac{(2G_E^V)^2 + \tau (2G_M^V)^2}{1 + \tau} \Delta
\]

\[
U_T^A = 2(1 + \tau)G_A^2 + G_A^2 \Delta
\]

\[
U_{T'} = 2G_A(2G_M^V)\sqrt{\tau(1 + \tau)[1 + \tilde{\Delta}]}
\]

with

\[
\tilde{\Delta} = \sqrt{\frac{\tau}{1 + \tau}} \frac{\xi_F(1 - \psi^2)}{2\kappa}
\]
Super-Scaling Analysis (SuSA)

Scaling in the RFG (Relativistic Fermi gas)

\[ R_K = G_K f_{RFG}(\psi) \]

Functions \( G_K \) from the RFG for electrons \( K = L, T \) and neutrinos \( K = CC, CL, LL, T, T' \).

Scaling function in the RFG

\[ f_{RFG}(\psi) = \frac{3}{4}(1 - \psi^2)\theta(1 - \psi^2) \]

Scaling variable:

\[ \psi = \frac{1}{\sqrt{\xi_F}} \frac{\lambda - \tau}{\sqrt{(1 + \lambda)\tau + \kappa \sqrt{\tau(1 + \tau)}}} \]
Experimental scaling function from \((e, e')\)

\[
f(\psi') = \frac{\frac{d\sigma}{d\Omega'd\epsilon'}}{\sigma_{\text{Mott}}(v_L G_L + v_T G_T)}
\]

shifted \(\rightarrow\) \[
\psi' = \frac{1}{\sqrt{\xi_F}} \frac{\lambda' - \tau'}{\sqrt{(1 + \lambda')\tau' + \kappa\sqrt{\tau'(1 + \tau')}}}
\]

\[
\lambda' = (\omega - E_s)/2m_N, \quad \tau' = \kappa^2 - \lambda'^2
\]

\(k_F\) and \(E_s\) are fitted to the data

\[
f_L = \frac{R_L}{G_L} \quad \text{Longitudinal} \quad f_T = \frac{R_T}{G_T} \quad \text{Transverse}
\]
Superscaling

- Plot the experimental $f(\psi')$ versus $\psi'$ for different kinematics and nuclei
- Fit $E_s$ and $k_F$ to get scaling (one universal scaling function)
Scaling in the QE peak

Scaling of $R_L$ [Donnelly & Sick PRC 60 (1999)]

FIG. 10. (Color) Scaling function $f_L(\psi')$ from the longitudinal response.
Fit in the Quasi-elastic peak

Scaling function
SuSA (Super Scaling Analysis)

- Using the experimental \((e, e')\) scaling function to predict neutrino cross sections
- Use the RFG equations to compute the \((\nu_l, l^-)\) response functions with the substitution \(f_{RFG}(\psi) \rightarrow f_{exp}(\psi)\)
Two-particle two-hole Meson Exchange Currents (MEC)

- Relativistic Fermi Gas two-nucleon emission channel
- Added to the SuSA results
- J.E. Amaro et al. PRC 82, 0444601 (2010)
2N emission in Fermi gas

- Initial state $|i\rangle = |F\rangle$,
- Sum over final states

$$\sum_f = \sum_{1p-1h} + \sum_{2p-2h} + \sum_{other channels}$$

- 2p-2h channel final states

$$|f\rangle = |2p-2h\rangle = |1', 2', 1^{-1}, 2^{-1}\rangle$$


$$|1\rangle = |h_1 s_1 t_1\rangle \quad |2\rangle = |h_2 s_2 t_2\rangle$$

$$|1'\rangle = |p_1' s_1' t_1'\rangle \quad |2'\rangle = |p_2' s_2' t_2'\rangle$$

- Particles $P_i' = (E_i', p_i')$, Pauli blocking $p_i' > k_F$
- Holes $H_i = (E_i, h_i)$, with $h_i < k_F$. 
2p-2h Hadronic tensor

\[ W^{\mu\nu} = W^{\mu\nu}_{1p1h} + W^{\mu\nu}_{2p2h} + \cdots \]

\[
W^{\mu\nu}_{2p2h} = \frac{V}{(2\pi)^9} \int d^3p_1' d^3p_2' d^3h_1 d^3h_2 \frac{m_N^4}{E_1 E_2 E'_1 E'_2} r^{\mu\nu}(p_1', p_2', h_1, h_2)
\]

\[ \times \delta^3(p_1' + p_2' - h_1 - h_2 - q) \delta(E'_1 + E'_2 - \omega - E_1 - E_2) \]

\[ \times \Theta(p_1', p_2', h_1, h_2) \]

\[ \Theta(p_1', p_2', h_1, h_2) = \theta(p_1' - k_F) \theta(p_2' - k_F) \theta(k_F - h_1) \theta(k_F - h_2) \]

Elementary hadronic tensor for a nucleon pair transition:

\[ r^{\mu\nu}(p_1', p_2', h_1, h_2) = \frac{1}{4} \sum_{s_1 s_2 s_1' s_2'} \sum_{t_1 t_2 t_1' t_2'} j^\mu(1', 2', 1, 2)_A^* j^\nu(1', 2', 1, 2)_A \]

\[ V = 3\pi^2 N/k_F^3. \]
Meson-Exchange Currents

Feynman diagrams:
Seagull (a,b), pionic (c), and \( \Delta \) current (d-g)

Pionic four-momenta
\[
K_{i}^{\mu} = P_{i}^{\mu} - H_{i}^{\mu}
\]
Seagull and pionic

Seagull:

\[ j_s^{\mu}(p_1', p_2', p_1, p_2) = \frac{f^2}{m_{\pi}^2} i \varepsilon_{ab} \bar{u}(p_1') \tau_a \gamma_5 \, K_1 \, u(p_1) \]

\[ \times \frac{F_V^1}{K_1^2 - m_{\pi}^2} \bar{u}(p_2') \tau_b \gamma_5 \gamma^{\mu} u(p_2) + (1 \leftrightarrow 2). \] (1)

Pion in flight:

\[ j_p^{\mu}(p_1', p_2', p_1, p_2) = \frac{f^2}{m_{\pi}^2} i \varepsilon_{ab} \frac{F_{\pi}(K_1 - K_2)^{\mu}}{(K_1^2 - m_{\pi}^2)(K_2^2 - m_{\pi}^2)} \]

\[ \times \bar{u}(p_1') \tau_a \gamma_5 \, K_1 \, u(p_1) \bar{u}(p_2') \tau_b \gamma_5 \, K_2 \, u(p_2). \] (2)

\( F_V^1 \) and \( F_{\pi} \): the electromagnetic form factors

pion-nucleon coupling constant: \( f^2/4\pi = 0.08 \).
\[ j_\Delta^\mu(p'_1, p'_2, p_1, p_2) = \frac{f_{\pi N} \Delta f}{m_\pi^2} \frac{1}{K_2^2 - m_\pi^2} \bar{u}(p'_1) T_\mu^\mu(1) u(p_1) \]

\[
\times \bar{u}(p'_2) \tau_\alpha \gamma_5 K_2 u(p_2) + (1 \leftrightarrow 2). \tag{3}
\]

\( T_\mu^\mu(1) \) is related to the pion electroproduction amplitude

\[
T_\mu^\mu(1) = K_{2,\alpha} \Theta^\alpha_\beta G_{\beta\rho}^\Delta (H_1 + Q) S_f^{\rho\mu} (H_1) T_a T_3^\dagger + T_3 T_\dagger a S_b^{\mu\rho} (P'_1) G_{\rho\beta}^\Delta (P'_1 - Q) \Theta_\beta^\alpha K_{2,\alpha}. \tag{4}
\]
The tensor $\Theta_{\mu\nu}$

$$\Theta_{\mu\nu} = g_{\mu\nu} - \frac{1}{4} \gamma_{\mu} \gamma_{\nu}.$$ (7)
**Δ propagator**

Rarita-Schwinger tensor

\[ G_{\beta\rho}^\Delta(P) = -\frac{P + m_\Delta}{P^2 - m_\Delta^2} \]

\[ \times \left[ g_{\beta\rho} - \frac{1}{3} \gamma_\beta \gamma_\rho - \frac{2}{3} \frac{P_\beta P_\rho}{m_\Delta^2} - \frac{\gamma_\beta P_\rho - \gamma_\rho P_\beta}{3m_\Delta} \right] . \]  (8)

**Δ width:** \( m_\Delta \rightarrow m_\Delta + \frac{i}{2} \Gamma(P) \) in the denominator of the propagator to account for the \( \Delta \) decay probability
Integration of the energy delta function

9-D integral for the 2p-2h response functions

$$
\int d^3 p'_1 d^3 h_1 d^3 h_2 \delta(E_1 + E_2 + \omega - E'_1 - E'_2)f(h_1, h_2, p'_1, p'_2),
$$

Momentum conservation $p'_2 = h_1 + h_2 + q - p'_1$.

We integrate over the momentum $p'_1$ using the delta function:

- For fixed $h_1, h_2, \theta'_1, \phi'_1$
- Change variables $p'_1 \rightarrow E' = E'_1 + E'_2$.
- Compute the Jacobian of the transformation

$$
dp'_1 = \frac{dE'}{|p'_1/E'_1 - p'_2 p'_1/E'_2 p'_1|},
$$
Momentum of the final nucleon

• Compute $p_1'$ for fixed angles $\theta_1'$, $\phi_1'$, by solving the energy conservation equation.

• Second degree equation with two solutions

$$p_1' = \frac{a}{b} \left( v \pm v_0 \sqrt{1 - \frac{b m_N^2}{a^2}} \right), \quad (11)$$

where

$$a = \frac{1}{2} \frac{p'^2}{b} \quad b = E'^2 - p'^2 \cos^2 \beta_1' \quad (12)$$

$$v_0 = E' \quad v = p' \cos \beta_1', \quad (13)$$

Final total energy: $E' = E_1 + E_2 + \omega$
Final total momentum: $p' = \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{q}$

$\beta_1' = \text{angle between } p_1' \text{ and } p'$. 
Results for 2p-2h responses

- Compute transverse response functions in the 2p-2h channel.
- Relativistic Fermi gas
- 7D integrals \( \int d^3 h_1 d^3 h_2 d\theta'_1 \)
- We choose \( \phi'_1 = 0 \) and multiply by \( 2\pi \).
- \( p'_1 \) is fixed from energy conservation
(e, e') results with MEC 2p-2h

- Quasielastic + MEC + Inelastic cross section
- $\epsilon_e = 680$ MeV
- $\theta = 60^\circ$
- Data from Saclay
(e, e') results with MEC 2p-2h

- Quasielastic + MEC + Inelastic cross section
- $\epsilon_e = 3595$ MeV
- $\theta = 16^o$
- Data from SLAC
(e, e') results with MEC 2p-2h

- Quasielastic + MEC + Inelastic cross section
- $\epsilon_e = 4045$ MeV
- $\theta = 60^\circ$
- Data JLab
2 Status of MEC in SuSA for $(\nu_\mu, \mu^{-})$


• Double differential neutrino cross sections from $^{12}$C
• Integrated over the neutrino flux
• Contribution of vector meson-exchange currents in the 2p-2h sector

The $R_{TT}^{VV}$ response with MEC is computed assuming that the weak CC MEC contribution is the same as the MEC contribution to the isovector part of the electromagnetic $R_T$

$$R_{TT}^{VV} = \left[R_{TT}^{VV}\right]_{OB} \left[R_{TT}^{em}\right]_{isovector} + \left[R_{TT}^{em}\right]_{2p2hMEC}$$
Neutrino results with MEC 2p-2h

- The MEC increase the cross section less than 10%
- Data from A.A. Aguilar-Arevalo et al., (MiniBooNE Collaboration), PRD 81, 092005 (2010)
Neutrino results. Angle projection

- The MEC tend to increase the cross section about 5-10%.
- Data from Aguilar-Arevalo et. al. (MiniBooNE Collaboration)
Antineutrino results with MEC 2p-2h

• Calculations from Amaro, Barbaro, Caballero, Donnelly, PRL 108 (2012).
• The MEC tend to increase the cross section more than for neutrinos.
• Data from Aguilar-Arevalo et. al. (MiniBooNE Collaboration) PRD 88 (2013)
Further modifications of SuSA

- Estimation of the axial 2p-2h MEC:

\[
\left[ R^{AA}_T \right]^{2p2h\text{MEC}} \simeq \left[ R^{VV}_T \right]^{2p2h\text{MEC}}
\]
\[
\left[ R^{VA}_T \right]^{2p2h\text{MEC}} \simeq \left[ R^{VV}_T \right]^{2p2h\text{MEC}}
\]

- SuSA v2: Two different scaling functions \( f_L \) and \( f_T \) (from the Relativistic Mean Field)

- Extend SuSA for low \( q \) and \( \omega \) with Pauli blocking

\[
 f_{PB}[\psi(\omega, q)] = f[\psi(\omega, q)] - f[\psi(-\omega, q)]
\]
**SuSA New results (preliminary)**

- Total cross section versus neutrino energy.
- SuSA with and without MEC

### Graph
- Total cross section $\sigma$ versus neutrino energy $E_{\nu}$.
- Various models and scenarios are compared, including SuSA with and without MEC, with and without particular configurations.

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**Nuint14**
**SuSA new results for MINERVA**

- MINERVA neutrino differential cross section
- SuSA with and without MEC

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**Graph Details**

- **Axes:**
  - **X-axis:** $Q_{QE}^2$ (GeV$^2$)
  - **Y-axis:** $d\sigma/dQ_{QE}^2$ [$10^{-39}$ cm$^2$/GeV$^2$/neutron]

- **Curves:**
  - RFG
  - SuSA + MEC ($T_{VV}$), newPB
  - SuSA, newPB
  - SuSAv2, newPB
  - SuSAv2 + MEC ($T_{VV}$), newPB
  - Minerva

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**Legend:**

- RFG
- SuSA + MEC ($T_{VV}$), newPB
- SuSA, newPB
- SuSAv2, newPB
- SuSAv2 + MEC ($T_{VV}$), newPB
- Minerva
SuSA estimation of axial MEC

- $\nu$ cross section.
- Preliminary results.
- For the 2p-2h, $R_{AA}^T = R_{VA}^{T'} = R_{VV}^T$.
- SuSA with and without MEC.
SuSA estimation of axial MEC

- $\bar{\nu}$ cross section.
- Preliminary results.
- For the 2p-2h,

$$R_{AA}^T = R_{VA}^{T'} = R_{VV}^T$$

- SuSA with and without MEC
**SuSA estimation of axial MEC**

- MiniBooNE double-differential cross section.
- Preliminary results for the 2p-2h MEC,
  \[ R_{AA}^T = R_{VA}^T = R_{VV}^T \]
- SuSA with and without MEC
3 Perspectives: new approach to 2p-2h

- I. Ruiz Simo, C. Albertus, J.E. Amaro, M.B. Barbaro, J.A. Caballero, T.W. Donnelly,
  Relativistic effects in two-particle emission for electron and neutrino reactions

See also:
- C. Albertus, et al.,
  2p-2h distribution in phase space for neutrino and electron scattering
  Nuint14 Poster P:01
- I. Ruiz Simo, et al.,
  Relativistic effects and meson exchange currents in two-particle emission with neutrinos.
  Nuint14 Poster P:15
New MEC calculation

- Including the axial current
- Codes more efficient and faster
- Relativistic effects
- Angular distribution of final nucleons: singularities for high $q$
- Analytical integration around the singularities

Study of the 7D integral in 2p-2h hadronic tensor

$$W_{2p-2h}^{\mu\nu} = \frac{V}{(2\pi)^9} \int d^3p_1'd^3h_1d^3h_2 \frac{m_N^4}{E_1E_2E'_1E'_2} \Theta(p'_1, p'_2, h_1, h_2)$$

$$r^{\mu\nu}(p'_1, p'_2, h_1, h_2) \delta(E'_1 + E'_2 - E_1 - E_2 - \omega)$$

$$p'_2 = h_1 + h_2 + q - p'_1.$$ 

$r^{\mu\nu} =$ elementary hadronic tensor for 2p-2h
Simplest case: \( r^{\mu \nu} = 1 \).

2p-2h phase-space function

\[
F(q, \omega) \equiv \int d^3 p_1' d^3 h_1 d^3 h_2 \frac{m_N^4}{E_1 E_2 E_1' E_2'} \delta(E_1' + E_2' - E_1 - E_2 - \omega) \Theta(p_1', p_2', h_1, h_2).
\]

with \( p_2' = h_1 + h_2 + q - p_1' \).

All 2p-2h models should agree about \( F(q, \omega) \).
Non-relativistic phase space

Semi-analytical

$$F'(q, \omega) = (2\pi)^3 \frac{k_F^7 m_N}{q_F} \int_0^{x_{\text{max}}} \frac{dx}{x^2} \int_{|q_F-x|}^{q_F+x} \frac{dy}{y^2} A(x, y, \nu),$$

$$x_{\text{max}} = 1 + \sqrt{2(1 + \nu)}, \quad q_F = \frac{q}{k_F} \quad \nu = \frac{m_N \omega}{k_F^2}.$$

Van Orden-Donnelly function:

$$A(l_1, l_2, \nu) = \frac{l_1^3 l_2^3}{(2\pi)^2} \int d^3x_1 d^3x_2 \delta(\nu - l_1 \cdot x_1 - l_2 \cdot x_2) \theta \left(1 - \left| x_1 - \frac{l_1}{2} \right| \right) \theta \left(1 - \left| x_2 - \frac{l_2}{2} \right| \right) \theta \left(\left| x_1 + \frac{l_1}{2} \right| - 1 \right) \theta \left(\left| x_2 + \frac{l_2}{2} \right| - 1 \right).$$
Van Orden function $A(x, y, \nu)$

- The function $A(x, y, \nu)$ is analytical (J.W. Van Orden, T.W. Donnelly, Ann. Phys. 131 (1981) 451)
- 3D plot for $q = 500$ MeV/c
Non-relativistic. Numerical

- $\phi'_1 = 0$
- Integrate over $p'_1$ for $h_1$, $h_2$ and $\theta'_1$ fixed.
- Sum over two solutions $p'_1(\pm)$ of the energy conservation equation.
- 7D integral

$$F(q, \omega) = 2\pi \int d^3h_1 d^3h_2 d\cos \theta'_1 \sum_{\alpha=\pm} \frac{p'_1^2 m_N}{|p'_1 - p'_2 \cdot \hat{p}'_1|} \Theta(p'_1, p'_2, h_1, h_2) \bigg|_{p'_1 = p'_1(\alpha)}$$

- Asymptotic expansion ($\sim \sqrt{m_N \omega}$)

$$F(q, \omega) \xrightarrow{\omega \to \infty} 4\pi \left(\frac{4}{3}\pi k_F^3\right)^2 \frac{m_N}{2} \sqrt{m_N \omega}.$$
Non-relativistic results

Comparison of phase space $F(q, \omega)$ calculations:

- Exact (semi-analytical)
- Numerical 7D integration with $10^7$ points
- Asymptotic
Relativistic phase-space

- 7D integral ($\phi'_1 = 0$)

\[ F(q, \omega) = 2\pi \int d^3h_1 d^3h_2 d\theta'_1 \sin\theta'_1 \frac{m_N^4}{E_1 E_2 E'_1 E'_2} \times \sum_{\alpha = \pm} \left| \frac{p'_1^2}{E_1'} - \frac{p'_2 \cdot \hat{p}'_1}{E'_2} \right| \Theta(p'_1, p'_2, h_1, h_2) \bigg|_{p'_1 = p'_1^{(\alpha)}} \]

the sum runs over the two solutions $p'_1^{(\pm)}$ of the relativistic energy conservation equation

- Asymptotic expansion $\sim m_N$ (constant!)

\[ F(q, \omega) \xrightarrow{\omega \to \infty} 4\pi \left( \frac{4}{3} \pi k_F^3 \right)^2 \frac{m_N^2}{2}. \]
Relativistic results.

- Straightforward integration.
- For low \( q \) and \( \omega \) the relativistic \( F(q, \omega) \) converge to the non-relativistic phase space.
- Numerical problems for high \( q \).
For high $q$ a spurious peak appears at low $\omega$ as a result of numerical error in the straightforward 7D integration.

We study the specific case $q = 3$ GeV/c.

- $q = 3$ GeV/c
- $q = 5$ GeV/c
For high \( q \gg k_F \), the hole momenta can be neglected, \( h_1 = h_2 = 0 \)

A 6-D integration can be performed analytically

\[
\bar{F}(q, \omega) = \int d^3 h_1 d^3 h_2 d^3 p'_1 \delta(E'_1 + E'_2 - \omega - 2m_N) \Theta(p'_1, p'_2, 0, 0) \frac{m^2_N}{E'_1 E'_2}
\]

\[
= \left( \frac{4}{3} \frac{\pi k_F^3}{k_F^3} \right)^2 \int d^3 p'_1 \delta(E'_1 + E'_2 - \omega - 2m_N) \Theta(p'_1, p'_2, 0, 0) \frac{m^2_N}{E'_1 E'_2}.
\]

The integral over \( \phi'_1 \) gives a factor \( 2\pi \), with \( \phi'_1 = 0 \)

The integral over \( p'_1 \) is done analytically

In the frozen nucleon approximation, the phase-space function is reduced to a 1D integral over \( \theta'_1 \)
Numerical error shown as discontinuities

$F(q, \omega)$ includes contributions from different $(h_1, h_2)$, with discontinuities at different $\omega$-points

The statistical distribution of millions of discontinuities appears as a smooth function $F(q, \omega)$ with a bump in the region where there are more discontinuities.
Angular distribution of ejected nucleons

$\Phi(\theta') = \sin \theta' \int p_1'^2 dp_1' \delta(E_1 + E_2 + \omega - E_1' - E_2')$

$\times \Theta(p_1', p_2', h_1, h_2) \frac{m_N^4}{E_1 E_2 E_1' E_2'}$

$= \sum_{\alpha=\pm} \sin \theta_1' \frac{m_N^4}{E_1 E_2 E_1' E_2'} \frac{p_1'^2}{p_1'^2 - \frac{p_2' \hat{p}_1'}{E_2}} \Theta(p_1', p_2', h_1, h_2) \bigg|_{p_1' = p_1'(\alpha)}$

$q = 3 \text{ GeV/c}$

Quasielastic peak at $\omega = 2200 \text{ MeV}$

The denominator is zero for some angles for each energy.
Kinematical analysis

At the minimum

\[ \frac{dE_{ex}}{dp'_1} = 0. \]

The Jacobian diverges at the minimum

\[ dp'_1 = \frac{dE_{ex}}{\left| \frac{dE_{ex}}{dp_1} \right|}, \]

In the frozen nucleon limit

\[ E_{ex} = E'_1 + E'_2 - E_1 - E_2. \]

\[ E_{ex} = \sqrt{p'_1^2 + m^2_N} + \sqrt{p'_2^2 + m^2_N + q^2 - 2p'_1q \cos \theta'_1} - 2m^2_N, \]
Analysis of the angular distribution

General form of integral over angles

\[ I = \int_{0}^{\pi} d\theta'_{1} \frac{f(\theta'_{1})}{\sqrt{g(\theta'_{1})}} \theta(g(\theta'_{1})) , \]

for positive values of the function

\[ g(\theta'_{1}) \equiv \cos^{2}(\theta'_{1} - \alpha) - w_{0}. \]

Non-dimensional variable

\[ w_{0} = \frac{E'_{2}}{s'_{2}} \left( 1 - \frac{(E'_{2}^{2} - p'_{2}^{2})^{2}}{4m_{N}^{2}E'^{2}} \right) \]

The integration region is determined by \( g(\theta'_{1}) > 0 \).

Three cases depending on \( w_{0} \):

- \( w_{0} > 1 \). The angular distribution is zero.
- \( w_{0} < 0 \). All angles allowed
- \( 0 \leq w_{0} \leq 1 \). The angular distribution is different from zero only in one or two angular intervals.

It is infinite for \( \cos^{2}(\theta'_{1} - \alpha) = w_{0} \)

\[ \Rightarrow \cos(\theta'_{1} - \alpha) = \pm \sqrt{w_{0}} \]

Position of the divergence:

\[ \theta'_{1} - \alpha = \varphi_{1} \pm \pi, \varphi_{2} \pm \pi . \]

with

\[ \varphi_{1} \equiv \cos^{-1} \sqrt{w_{0}}, \quad \varphi_{2} \equiv \cos^{-1}(-\sqrt{w_{0}}), \]

\[ 0 \leq \varphi_{1}, \varphi_{2} < \pi. \]
Allowed angular intervals

\[ 0 \leq w_0 \leq 1 \]

- Exact position of the divergence and the intervals.
- Eight possible cases
- Classified according to the values of \( \alpha \) and \( w_0 \).
Integration of divergences

- The divergence is integrable.
- Similar to
  \[ \int_0^\epsilon \frac{dx}{\sqrt{x}} = 2\sqrt{x}\bigg|_0^\epsilon = 2\sqrt{\epsilon} \]

\[ I(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \frac{f(\theta)d\theta}{\sqrt{g(\theta)}} \]
\[ = I(\theta_1, \theta_1 + \epsilon) + I(\theta_1 + \epsilon, \theta_2 - \epsilon) + I(\theta_2 - \epsilon, \theta_2) . \]

- Semi-analytical integration
  \[ I(\theta_1, \theta_1 + \epsilon) = \int_{\theta_1}^{\theta_1+\epsilon} \frac{f(\theta)d\theta}{\sqrt{g(\theta)}} \approx 2 \frac{f(\theta_1)}{g'(\theta_1)} \int_{\theta_1}^{\theta_1+\epsilon} \frac{d\sqrt{g(\theta)}}{d\theta} d\theta \]
\[ = 2 \frac{f(\theta_1)}{g'(\theta_1)} \sqrt{g(\theta_1 + \epsilon)} \approx \frac{f(\theta_1) \sqrt{2\epsilon}}{[w_0(1 - w_0)]^{1/4}} \]
Phase space. New integration method.

- Results do not depend on $\epsilon$
- Number of $\theta_1'$ points: $n = 7$
- Number of $h_1, h_2$ points: $5^6$
- Total number of 7D points: $5^6 \times 7 \approx 10^5$
Frozen nucleon approximation

- The frozen approximation is good for moderate to high momentum transfer except for very low energy tail.
Average momentum approximation

- The hole momenta $h_1, h_2$ are set to a constant inside the integral.
- $F(q, \omega)$ is the average over all the $h_1, h_2$ configurations.
- Pairs of configurations with opposite total momentum $p = h_1 + h_2$ average to the frozen nucleon approximation.
- Check with parallel, anti-parallel and perpendicular configurations.
Parallel and anti-parallel configurations

Why the frozen approximation works?

- UU vs. DD average to the frozen approximation
- UD and $T, -T$ are pairs with high relative momentum, like correlated nucleons. They contribute the same as the frozen approximation because the total momentum is zero.
Application to 2p-2h MEC in neutrino reactions

Weak CC Seagull operator

\[ j^\mu_s(p_1', p_2', h_1, h_2) = [\tau_0 \otimes \tau_{+1} - \tau_{+1} \otimes \tau_0] J^\mu(p_1', p_2', h_1, h_2), \]

with vector and axial currents

\[ J^\mu(p_1', p_2', h_1, h_2) = \frac{f}{m_\pi \sqrt{2} f_\pi} \bar{u}(p_2') \left\{ g_A F_1^V(Q^2) \gamma_5 \gamma^\mu + F_\rho(K_2^2) \gamma^\mu \right\} u(h_2) \]

\[ \frac{K_1 u(h_1)}{K_1^2 - m_\pi^2} \]

\[ - (1 \leftrightarrow 2) \]
**CC response functions**

Fully relativistic $R_L$ and $R_T$ compared to the OB 1p-1h RFG

![Graphs showing CC response functions for different momenta](image-url)
Relativistic effects are small in $R_T$ because $F_1^V(Q^2)$ is small where relativistic effects are large.
Isospin channels

- $PP$ emission dominates over $PN$
- The difference is due to the interference direct-exchange (D-X)
- D-X interference diagrams are NOT negligible
The frozen approximation is good for the seagull current and for the (D-X) interference diagrams.
Summary

- Optimization of the 7D integral in 2p-2h response functions
- Phase space function $F(q, \omega)$
- Angular distribution in the frozen approximation has divergencies for some angles
- Found the allowed angular regions and integrate analytically around the divergencies
- CPU time reduced by 100
- Relativistic results converge to the non-relativistic ones
- Test for electron and neutrino reactions with the contact operator.
- Frozen approximation (1D) very close to the exact (7D) results.
- We are working in the implementation of a complete set of MEC operators, including the axial part.
THANK YOU